# Chap 7: Trajectory Generation

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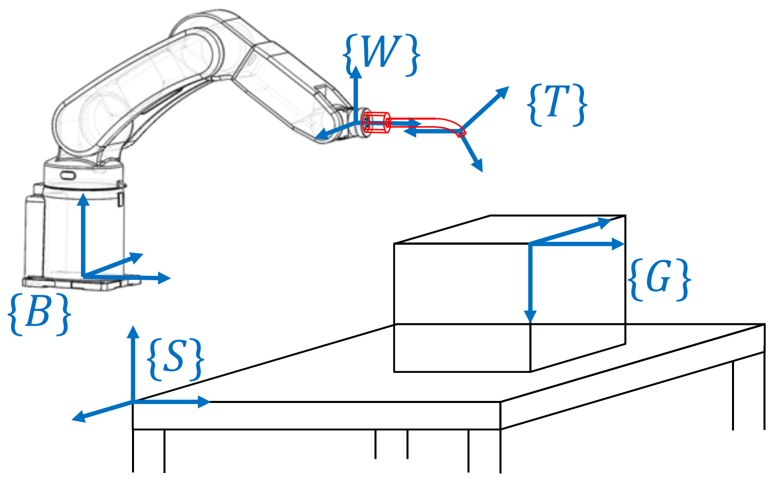
## Trajectory -1

- Definition: A time history of position, velocity, and acceleration of the manipulator
- In most cases: Considering {T} w.r.t. {G}

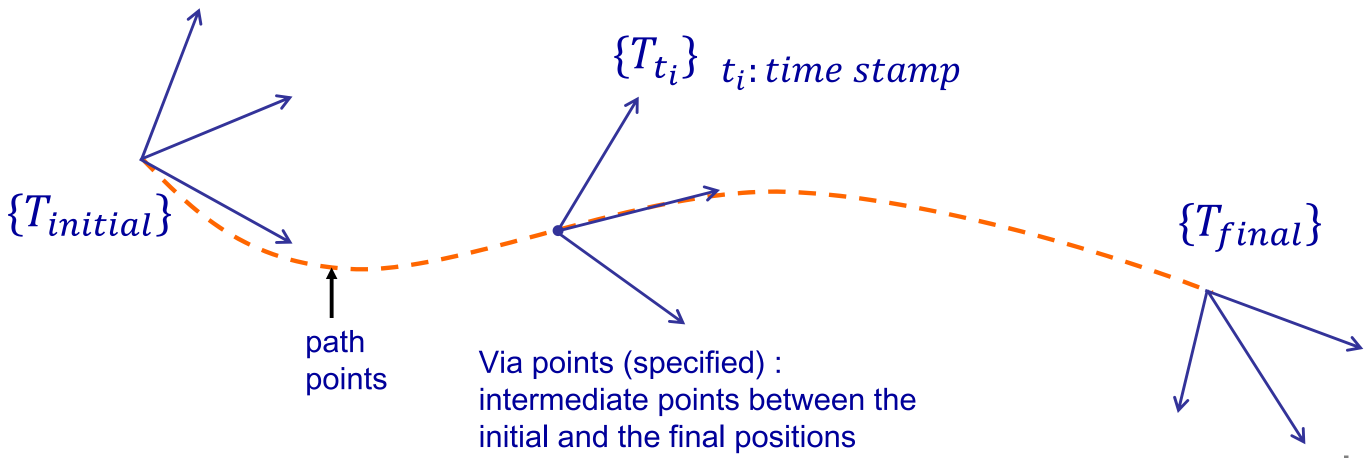
Independent of type of robot in use (↑ {T}) Can change with time; Ex: Conveyer (↑ {G})



# Trajectory -2

- Desire trajectory: Smooth path (i.e., continuous with continuous first derivative

usually ${}^{G}_{T}T$






# Path Generation in Joint-space -1

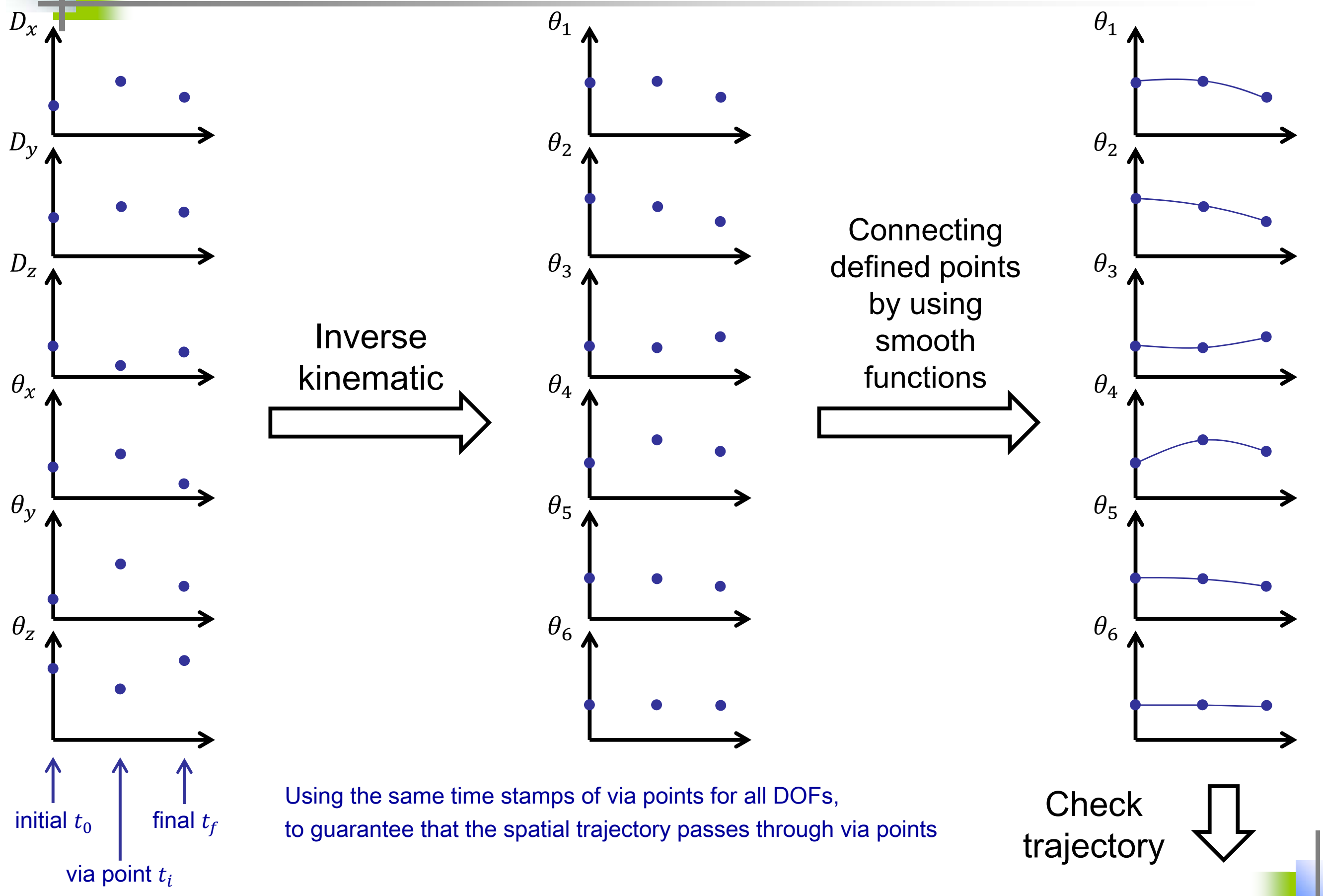
- Steps
  - Define initial, via, & final points of {T} w.r.t. to {G}, ${}^{G}_{T}T_i$
    (For both translational & rotational states)

    i=1 initial
    i=2~N-1 via points
    i=N+1 final

    represent ${}^{G}_{T}T_i$ in ${}^{G}X_T = \begin{bmatrix} {}^{G}P_{T\,org} \\ ROT({}^{G}\hat{K}_T, \theta) \end{bmatrix}$

    Not in rotation matrix
  - Inverse kinematics: $\{T_i\} \rightarrow \Theta_i$
  - Plan smooth trajectories for all DOFs (joints)
  - Forward Kinematics: Check generated path in Cartesian space

# Path Generation in Joint-space -2




Using the same time stamps of via points for all DOFs,
to guarantee that the spatial trajectory passes through via points



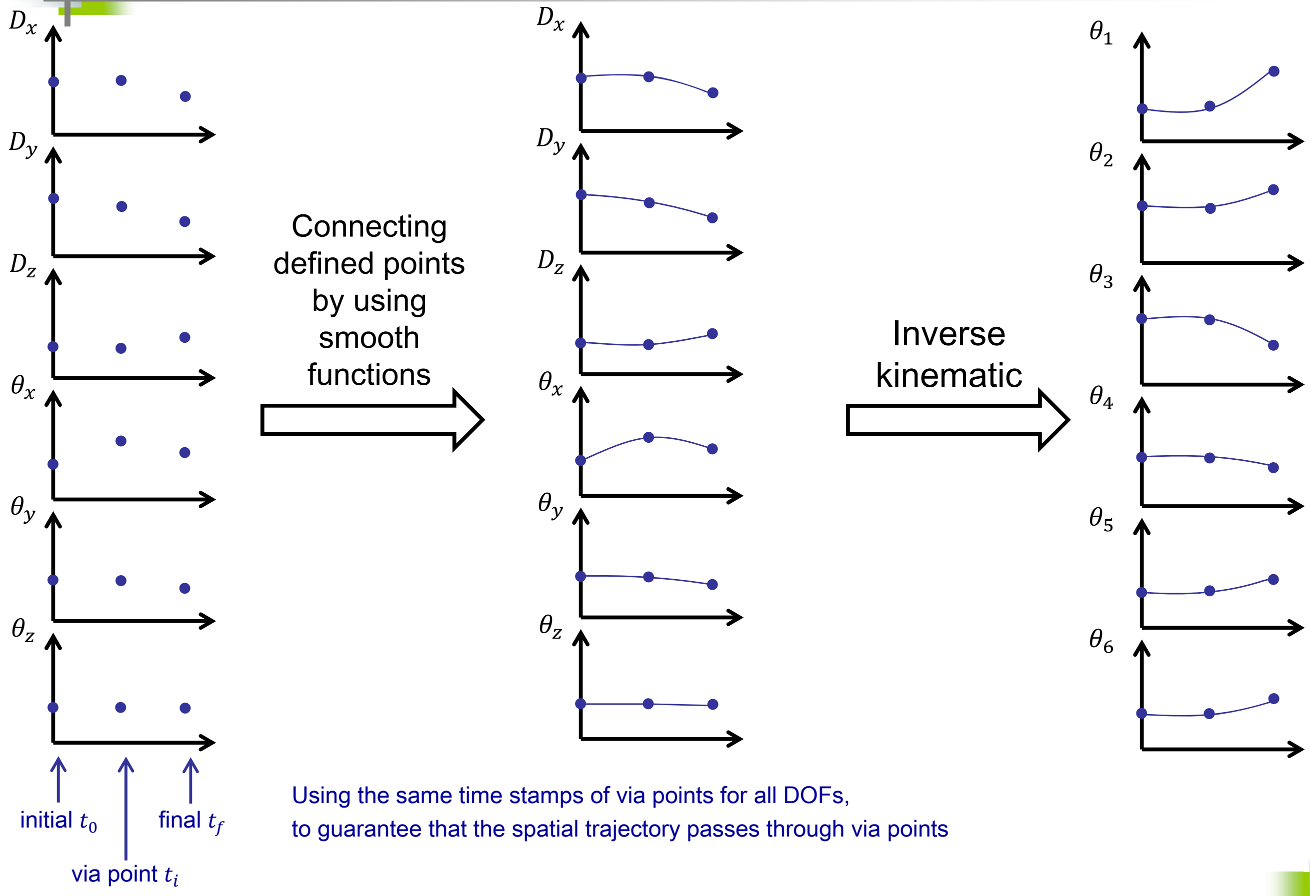
# Path Generation in Cartesian-space -1

- Steps
  - Define initial, via, & final points of {T} w.r.t. to {G}, ${}^{G}_{T}T_i$
    (For both translational & rotational states)
  - Plan smooth trajectories for all DOFs (joints)
  - Inverse kinematics: $\{T_i\} \rightarrow \Theta_i$

i=1 initial
i=2~N-1 via points
i=N+1 final

- Comments
  - Physically meaningful paths
  - Heavier computation load (i.e., IK)

# Path Generation in Cartesian-space -2



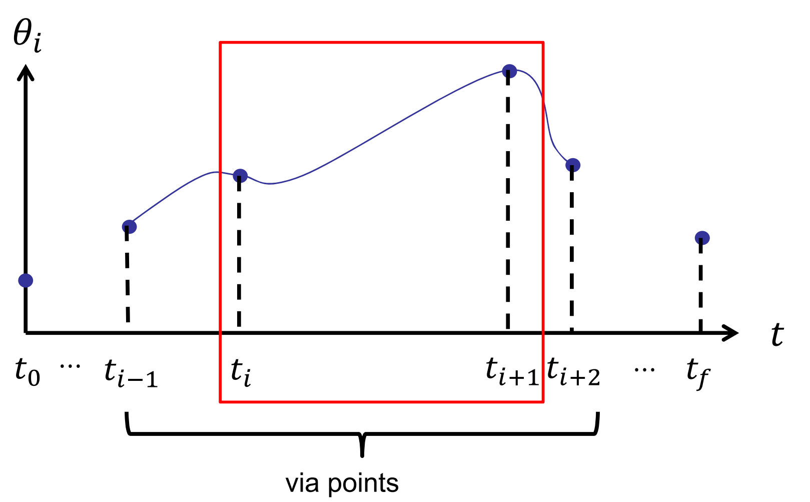

Using the same time stamps of via points for all DOFs,
to guarantee that the spatial trajectory passes through via points



# Cubic Polynomials -1

- Trajectory
  - Different sections $[t_i \quad t_{i+1}]$ use different functions
  - Smooth: Need to specify $\theta(t_i), \theta(t_{i+1}), \dot{\theta}\,(t_i), \dot{\theta}\,(t_{i+1})$

4 variables: need cubic polynomials




1 linear
2 quadratic
3 cubic
4 quartic
5 quintic
6 hexic(sextic)
7 heptic(septic)
8 octic
9 nonic
10 decic

# Cubic Polynomials -2

- Solve cubic polynomials
  - General form

$$\theta(\tilde{t}) = a_0 + a_1\tilde{t} + a_2\tilde{t}^2 + a_3\tilde{t}^3 \qquad 4\ unknowns:\ a_j \quad j = 0\sim 3$$

  - For each section $t \in [t_i, t_{i+1}]$

$$\tilde{t} = t - t_i \qquad so\ \tilde{t}|_{t=t_i} = 0 \text{ and } \tilde{t}|_{t=t_{i+1}} \equiv \Delta t = t_{i+1} - t_i$$

$\Delta t$ can be different for different $[t_i, t_{i+1}]$

Boundary conditions

$$\theta(\tilde{t}|_{t=t_i}) = \theta_i = a_0 \quad ①$$

$$\theta(\tilde{t}|_{t=t_{i+1}}) = \theta_{i+1} = a_0 + a_1\Delta t + a_2\Delta t^2 + a_3\Delta t^3 \quad ②$$

$$\dot{\theta}(\tilde{t}|_{t=t_i}) = \dot{\theta}_i = a_1 \quad ③$$

$$\dot{\theta}(\tilde{t}|_{t=t_{i+1}}) = \dot{\theta}_{i+1} = a_1 + 2a_2\Delta t + 3a_3\Delta t^2 \quad ④$$



# Cubic Polynomials -3

- Solving linear equations
  - By using ① ② ③ ④

$$a_2 = \frac{3}{\Delta t^2}(\theta_{i+1} - \theta_i) - \frac{2}{\Delta t}\dot{\theta}_i - \frac{1}{\Delta t}\dot{\theta}_{i+1}$$

$$a_3 = -\frac{2}{\Delta t^3}(\theta_{i+1} - \theta_i) + \frac{1}{\Delta t^2}(\dot{\theta}_{i+1} + \dot{\theta}_i)$$

# Cubic Polynomials -4

- Matrix operation

$$\begin{bmatrix} \theta_i \\ \theta_{i+1} \\ \dot{\theta}_i \\ \dot{\theta}_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & \Delta t & \Delta t^2 & \Delta t^3 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2\Delta t & 3\Delta t^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\Theta = T_{4\times4} \cdot A$$

$$\det(T_{4\times4}) = -\Delta t^4 \neq 0 \; as\ long\ as\ \Delta t \neq 0)$$

Thus,

$$\mathrm{A} = T_{4\times4}^{-1}\Theta$$

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\frac{3}{\Delta t^2} & \frac{3}{\Delta t^2} & -\frac{2}{\Delta t} & -\frac{1}{\Delta t} \\ \frac{2}{\Delta t^3} & -\frac{2}{\Delta t^2} & \frac{1}{\Delta t^2} & \frac{1}{\Delta t^2} \end{bmatrix} \begin{bmatrix} \theta_i \\ \theta_{i+1} \\ \dot{\theta}_i \\ \dot{\theta}_{i+1} \end{bmatrix}$$



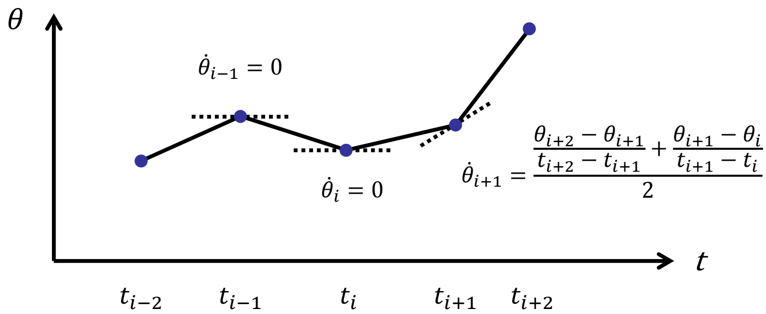
# Cubic Polynomials -5

- How to choose velocities $\dot{\theta}_i$ and $\dot{\theta}_{i+1}$?
  - User defined velocities in either Cartesian space or joint space

    NOT Recommend---Complicated, especially around singular points
  - Automatically generated

    Ex: Choose $\dot{\theta}_i = 0$ if $\dot{\theta}_i$ changes sign before/after $t_i$

    Choose *average* if not




The Cubic polynomials from different sections can be solved separately

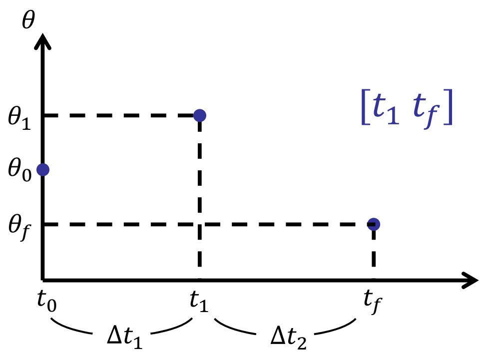
# Cubic Polynomials -6

- Set velocities such that the acceleration is CONTINUOUS

  (Use this free tuning variable nicely! )

  The Cubic polynomials from different sections should be solved simultaneously

- Example: A trajectory with one via point




$[t_0\ t_1]$ $\quad \Delta t_1 = t_1 - t_0$

$$\theta_I(\tilde{t}) = a_{10} + a_{11}\tilde{t} + a_{12}\tilde{t}^2 + a_{13}\tilde{t}^3$$

$[t_1\ t_f]$ $\quad \Delta t_2 = t_f - t_1$

$$\theta_{II}(t) = a_{20} + a_{21}\tilde{t} + a_{22}\tilde{t}^2 + a_{23}\tilde{t}^3$$

$$\Rightarrow 8\ unknowns$$



# Cubic Polynomials -7

- Example: A trajectory with one via point (cont.)

4 position B.C.s
2 for each $\theta_j(t)$ $\ j = \mathrm{I}, \mathrm{II}$

$$\begin{cases} \theta_0 = a_{10} \\ \theta_1 = a_{10} + a_{11}\Delta t_1 + a_{12}\Delta t_1^2 + a_{13}\Delta t_1^3 = a_{20} \\ \theta_f = a_{20} + a_{21}\Delta t_2 + a_{22}\Delta t_2^2 + a_{23}\Delta t_2^3 \end{cases}$$

2 velocity B.C.s

$$\begin{cases} \dot{\theta}_0 = \boxed{0} = a_{11} \\ \dot{\theta}_f = \boxed{0} = a_{21} + 2a_{22}\Delta t_2 + 3a_{23}\Delta t_2^2 \end{cases}$$

*not necessary* "0"

Middle point
velocity continuity
acceleration continuity

$$\begin{cases} \dot{\theta}_1 = a_{11} + 2a_{12}\Delta t_1 + 3a_{13}\Delta t_1^2 = a_{21} \\ \ddot{\theta}_1 = 2a_{12} + 6a_{13}\Delta t_1 = 2a_{22} \end{cases}$$

$$\Rightarrow 8\ equations$$

## Cubic Polynomials -8

- Example: A trajectory with one via point (cont.)

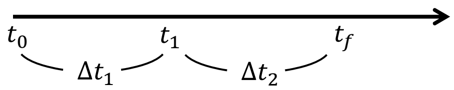
8 equations, 8 unknowns

Solution (when $\Delta t_1 = \Delta t_2 = \Delta t$)

$$a_{10} = \theta_0$$
$$a_{11} = 0$$
$$a_{12} = \frac{12\theta_1 - 3\theta_f - 9\theta_0}{4\Delta t^2}$$
$$a_{13} = \frac{-8\theta_1 + 3\theta_f + 5\theta_0}{4\Delta t^3}$$

$$a_{20} = \theta_1$$
$$a_{21} = \frac{3\theta_f - 3\theta_0}{4\Delta t}$$
$$a_{22} = \frac{-12\theta_1 + 6\theta_f + 6\theta_0}{4\Delta t^2}$$
$$a_{23} = \frac{8\theta_1 - 5\theta_f - 3\theta_0}{4\Delta t^3}$$

$$\Theta_{8\times1} = T_{8\times8}A_{8\times1}$$



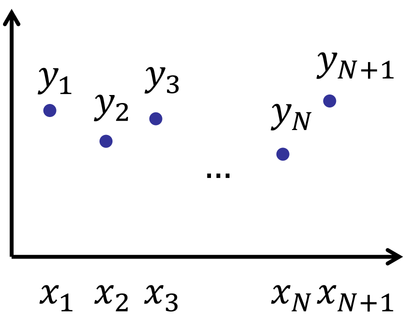

$$\det(T_{8\times8}) = 4\Delta t_1{}^4\Delta t_2{}^3 + 4\Delta t_1{}^3\Delta t_2{}^4$$
$$\neq 0 \; as\ long\ as\ \Delta t_1 \neq 0, \Delta t_2 \neq 0, \Delta t_1 \neq -\Delta t_2)$$



## Cubic Polynomials -9

- General cubic spline function

N+1 set points $(x_i, y_i)$
- 1 initial
- N-1 via
- 1 final




N cubic functions

$$s_j(x) = a_j + b_j x + c_j x^2 + d_j x^3 \qquad x_j \leq x \leq x_{j+1}, \quad j = 1 \dots N$$

$\Rightarrow$ total 4N unknown coefficients

## Cubic Polynomials -10

Position conditions at both ends of each $s_j(x)$

$\Rightarrow$ 2N conditions

Velocity & acceleration continuity conditions at via points

$\Rightarrow$ 2(N-1) conditions

NEED TWO MORE CONDITIONS for unique solution

Revisit example: A trajectory with one via point

$$\begin{bmatrix} y_1 = s_1(x_1) \\ y_2 = s_1(x_2) \end{bmatrix} \quad \begin{bmatrix} y_2 = s_2(x_2) \\ y_3 = s_2(x_3) \end{bmatrix}$$

$$\dot{y}_2 = \dot{s}_1(x_2) = \dot{s}_2(x_2)$$
$$\ddot{y}_2 = \ddot{s}_1(x_2) = \ddot{s}_2(x_2)$$

6 conditions



## Cubic Polynomials -11

Choices for the last 2 conditions:

(1) $\ddot{s}_1(x_1) = \ddot{s}_N(x_{N+1}) = 0$ — Natural cubic spline

(2) $\dot{s}_1(x_1) = u \quad \dot{s}_N(x_{N+1}) = v$ — Clamped cubic spline

(3) $if\ s_1(x_1) = s_N(x_{N+1})$ — Periodic cubic spline

$use\ \dot{s}_1(x_1) = \dot{s}_N(x_{N+1})$

$\ddot{s}_1(x_1) = \ddot{s}_N(x_{N+1})$

Note: Matlab® command *spline*

$$[YY] = spline(x, y, XX)$$

## High-order Polynomials

- If position, velocity and acceleration are all needed to be specified

$\Rightarrow$ Quintic polynomial $\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5 = \sum_{i=0}^{5} a_i t^i$

$$\begin{cases} \theta_0 = a_0 \\ \theta_f = a_0 + a_1\Delta t + a_2\Delta t^2 + a_3\Delta t^3 + a_4\Delta t^4 + a_5\Delta t^5 \\ \dot{\theta}_0 = a_1 \\ \dot{\theta}_f = a_1 + 2a_2\Delta t + 3a_3\Delta t^2 + 4a_4\Delta t^3 + 5a_5\Delta t^4 \\ \ddot{\theta}_0 = 2a_2 \\ \ddot{\theta}_f = 2a_2 + 6a_3\Delta t + 12a_4\Delta t^2 + 20a_5\Delta t^3 \end{cases}$$

$$a_0 = \theta_0 \qquad a_3 = \frac{20(\theta_f - \theta_0) - (8\dot{\theta}_f + 12\dot{\theta}_0)\Delta t - (3\ddot{\theta}_0 - \ddot{\theta}_f)\Delta t^2}{2\Delta t^3}$$

$$a_1 = \dot{\theta}_0 \qquad a_4 = \frac{30(\theta_0 - \theta_f) + (14\dot{\theta}_f + 16\dot{\theta}_0)\Delta t - (3\ddot{\theta}_0 - 2\ddot{\theta}_f)\Delta t^2}{2\Delta t^4}$$

$$a_2 = \frac{1}{2}\ddot{\theta}_0 \qquad a_5 = \frac{12(\theta_f - \theta_0) - (6\dot{\theta}_f + 6\dot{\theta}_0)\Delta t - (\ddot{\theta}_0 - \ddot{\theta}_f)\Delta t^2}{2\Delta t^5}$$

1 linear
2 quadratic
3 cubic
4 quartic
5 quintic
6 hexic(sextic)
7 heptic(septic)
8 octic
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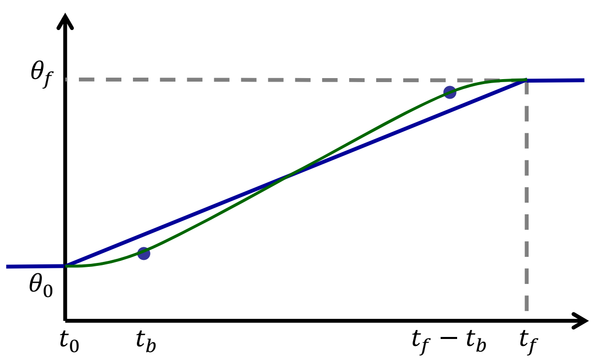
10 decic



## Linear Function with Parabolic Blends -1

- Setup
  - If only linear function is used -> velocity discontinuity
  - Solution: Modify both ends with parabolic functions to generate smooth velocity profiles

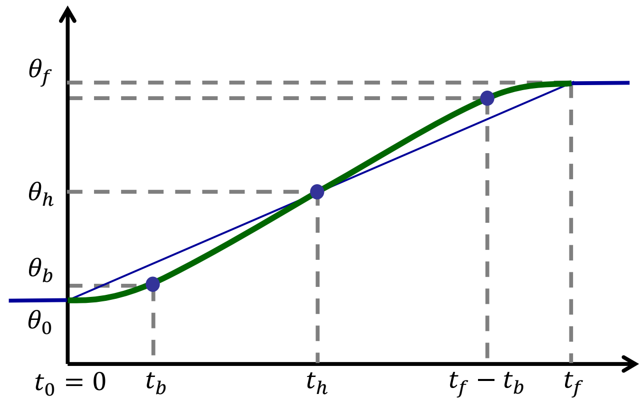



$(assume\ \dot{\theta}_0 = \dot{\theta}_f = 0)$

# Linear Function with Parabolic Blends -2

- Formulation
  - Linear section
    - Constant velocity

$$\dot{\theta} = \frac{\theta_h - \theta_b}{t_h - t_b} = \dot{\theta}_{t_b} \quad \dots ①$$




$t_h = \frac{1}{2} t_f \quad \theta_h = \frac{\theta_f + \theta_0}{2}$

$assume\ \dot{\theta}_0 = \dot{\theta}_f = 0$

  - Quadratic section

$$\theta(t) = \theta_0 + \dot{\theta}_0 t + \frac{1}{2}\ddot{\theta} t^2$$

acceleration (↑ $\ddot{\theta}$)

$$\dot{\theta}(t) = \dot{\theta}_0 + \ddot{\theta} t$$

$$\dot{\theta}(t_b) = \ddot{\theta} t_b \quad \dots ②$$



# Linear Function with Parabolic Blends -3

- Because ① = ②

$$\ddot{\theta} t_b = \dot{\theta}_{t_b} = \frac{\theta_h - \theta_b}{t_h - t_b} = \frac{\frac{\theta_f + \theta_0}{2} - (\theta_0 + \frac{1}{2}\ddot{\theta} t_b^2)}{\frac{t_f}{2} - t_b} = \frac{\theta_f - \theta_0 - \ddot{\theta} t_b^2}{t_f - 2t_b}$$

$$\ddot{\theta} t_b^2 - \ddot{\theta} t_f t_b + (\theta_f - \theta_0) = 0$$

$$\Rightarrow \quad t_b = \frac{\ddot{\theta} t_f - \sqrt{\ddot{\theta}^2 t_f^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}}$$

Need $\ddot{\theta} \geq \frac{4(\theta_f - \theta_0)}{t_f^2}$ to have meaningful solution

$$\ddot{\theta}_{min} \triangleq \frac{4(\theta_f - \theta_0)}{t_f^2}$$

## Linear Function with Parabolic Blends -4

- If $\ddot{\theta} = \ddot{\theta}_{min}$

$$t_b = \frac{t_f}{2} = t_h \quad \Rightarrow$$

No linear function,
two parabolic functions connecting to each other

at $t_b$

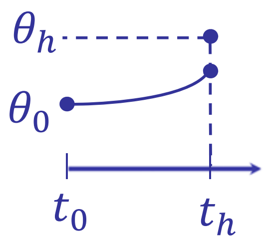
$$\dot{\theta}(t_b) = \ddot{\theta} t_b = \frac{4(\theta_f - \theta_0)}{{t_f}^2}\frac{t_f}{2} = 2\frac{\theta_f - \theta_0}{t_f}$$

Twice the velocity comparing to "linear only" situation

$$\dot{\theta} = \frac{\theta_f - \theta_0}{t_f}$$

- If $\ddot{\theta} < \ddot{\theta}_{min}$
  - Acceleration is not enough

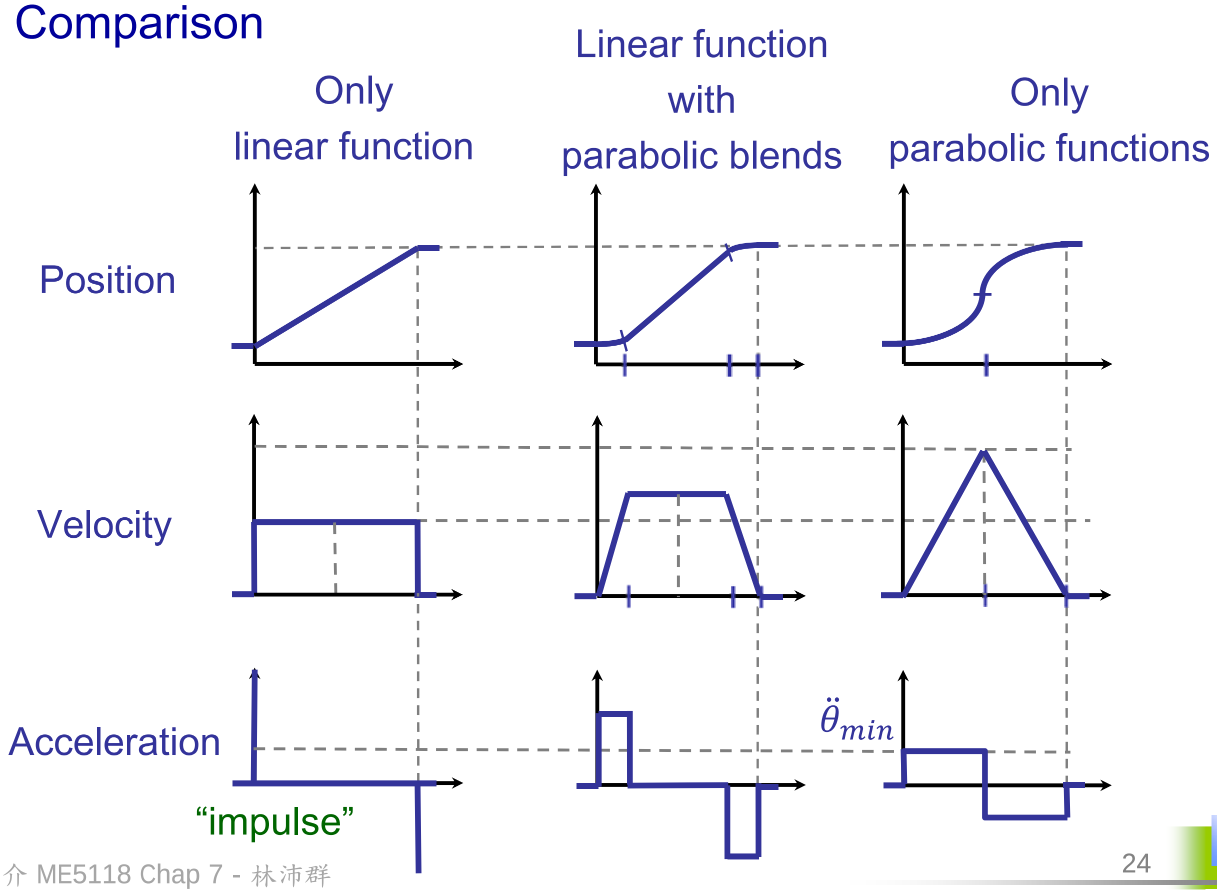
at $t_b = t_h, \quad \theta < \theta_h$





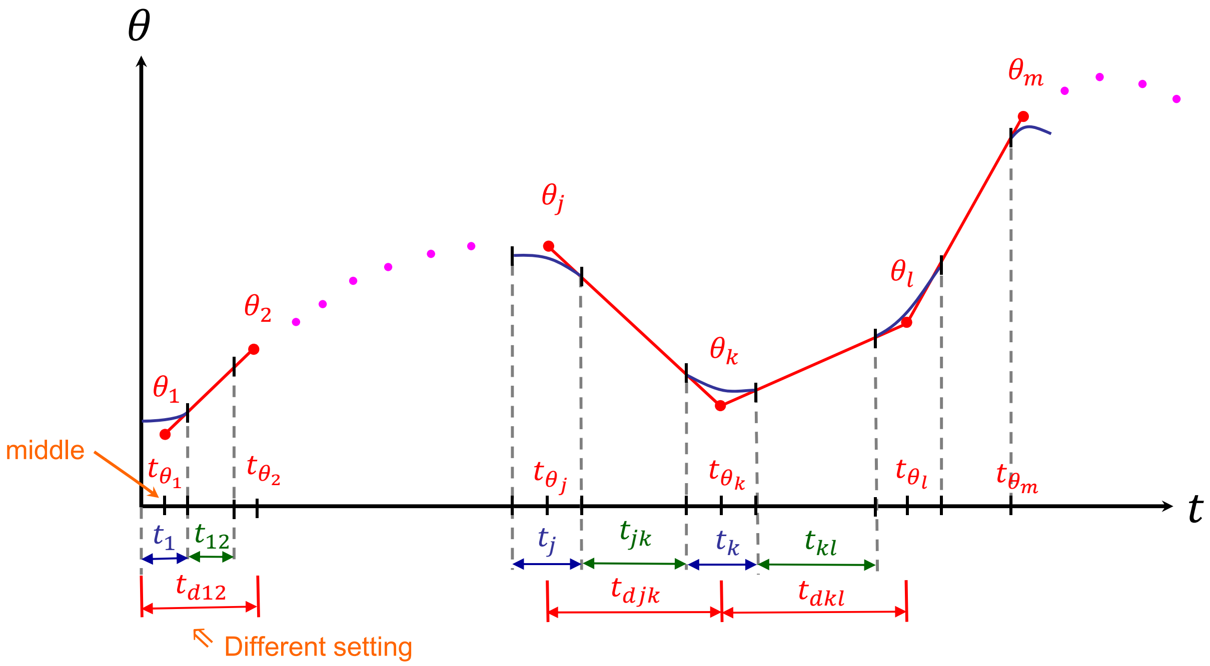

## Linear Function with Parabolic Blends -5

- Comparison



# Linear Function with Parabolic Blends -6

- General case: A path with $n$ via points
  - The same "linear sections" --- regard $[\ \theta_i\ \theta_{i+1}\ ]$ as "$[\ \theta_0\ \theta_f\ ]$"






# Linear Function with Parabolic Blends -7

- Middle segments $[\ \theta_i\ \theta_{i+1}\ ]$
  - Linear section

$$\dot{\theta}_{jk} = \frac{\theta_k - \theta_j}{t_{djk}}$$

$$\dot{\theta}_{kl} = \frac{\theta_l - \theta_k}{t_{dkl}}$$

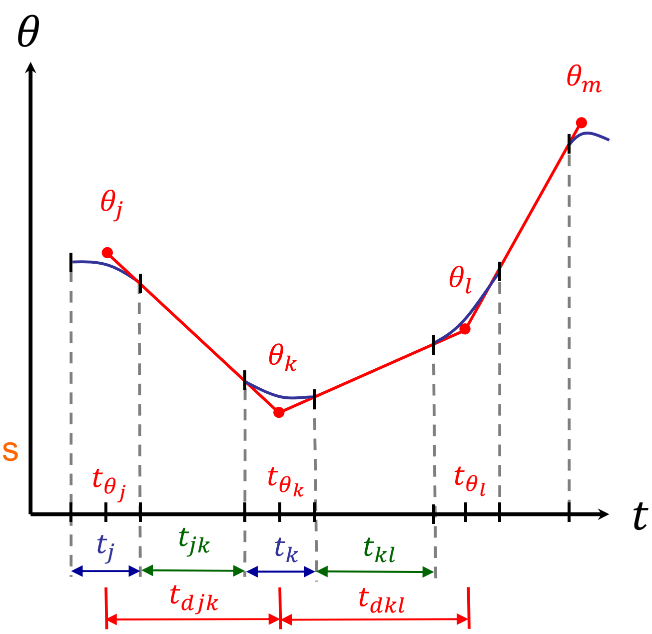
  - Parabolic section

$$\ddot{\theta}_k = sgn(\dot{\theta}_{kl} - \dot{\theta}_{jk})|\ddot{\theta}_k|$$

define this

$$t_k = \frac{\dot{\theta}_{kl} - \dot{\theta}_{jk}}{\ddot{\theta}_k}$$

$$t_{jk} = t_{djk} - \frac{1}{2}t_j - \frac{1}{2}t_k$$



# Linear Function with Parabolic Blends -8

- The first segment

$$\dot{\theta}_{12} = \frac{\theta_2 - \theta_1}{t_{d12} - \frac{1}{2}t_1} = \ddot{\theta}_1 t_1$$

Note: Because $t_1 = 2t_{\theta_1}$, $t_{\theta_1}$ should be close to $t_0 = 0$

$$\ddot{\theta}_1 = sgn(\theta_2 - \theta_1)|\ddot{\theta}_1|$$

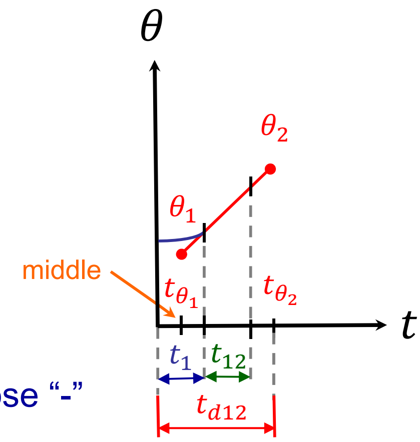
define this

Solve $t_1$

$$\frac{1}{2}\ddot{\theta}_1 t_1^2 - t_{d12}\ddot{\theta}_1 t_1 + (\theta_2 - \theta_1) = 0$$

$$t_1 = t_{d12} \pm \sqrt{{t_{d12}}^2 - \frac{2(\theta_2 - \theta_1)}{\ddot{\theta}_1}}$$

Choose "-"




$$t_{12} = t_{d12} - t_1 - \frac{1}{2}t_2$$



# Linear Function with Parabolic Blends -9

- The last segment, similar to the first segment

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n - \theta_{n-1}}{t_{d(n-1)n} - \frac{1}{2}t_n} = \ddot{\theta}_n(-t_n)$$

Note: Because $t_n = 2(t_f - t_{\theta_n})$, $t_{\theta_n}$ should be close to $t_f$

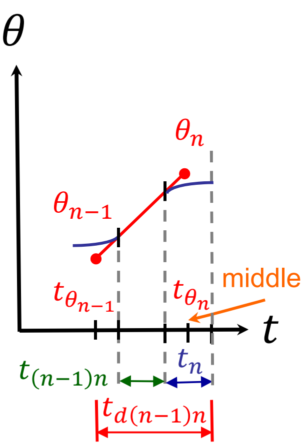
$$\ddot{\theta}_n = sgn(\theta_n - \theta_{n-1})|\ddot{\theta}_n|$$

define this

Solve $t_n$

$$\frac{1}{2}\ddot{\theta}_n t_n^2 - t_{d(n-1)n}\ddot{\theta}_n t_n + (\theta_n - \theta_{n-1}) = 0$$

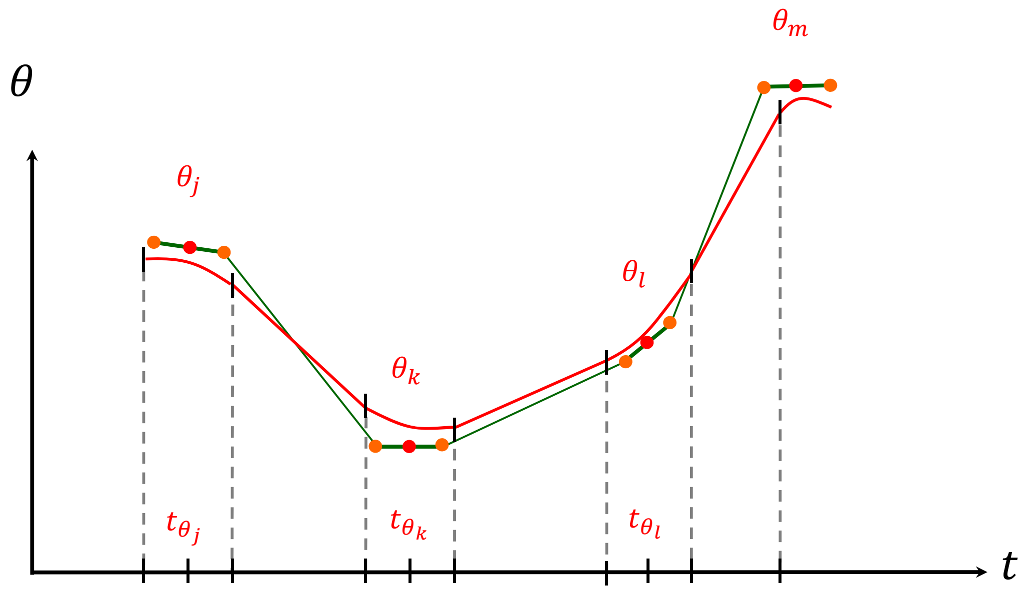
$$t_n = t_{d(n-1)n} - \sqrt{{t_{d(n-1)n}}^2 - \frac{2(\theta_n - \theta_{n-1})}{\ddot{\theta}_n}}$$




$$t_{(n-1)n} = t_{d(n-1)n} - t_n - \frac{1}{2}t_{n-1}$$

# Linear Function with Parabolic Blends -10

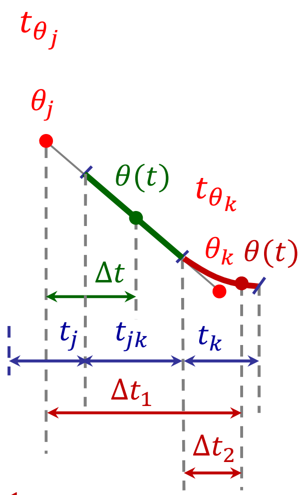
- Comments
  - Via points are not actually reached (i.e., reached when accel. → ∞)
  - If passing the via points are required ⇒ Use "pseudo via points"






# Linear Function with Parabolic Blends -11

- Straight line in joint space ⇒ Not necessary in Cartesian space
- In programming,
  - Check $t$ is in which line/parabolic section, and then use the correct linear/parabolic function




Ex: linear: $t \in [\, t_{\theta_j} + \frac{1}{2}t_j \quad t_{\theta_k} - \frac{1}{2}t_k \,]$

$$\theta(t) = \theta_j + \dot{\theta}_{jk}\Delta t = \theta_j + \dot{\theta}_{jk}(t - t_{\theta_j})$$

$$\dot{\theta}(t) = \dot{\theta}_{jk} \qquad \ddot{\theta} = 0$$

parabolic: $t \in [\, t_{\theta_k} - \frac{1}{2}t_k \quad t_{\theta_k} + \frac{1}{2}t_k \,]$

$$\theta(t) = \theta_j + \dot{\theta}_{jk}\Delta t_1 + \frac{1}{2}\ddot{\theta}_k\, \Delta t_2{}^2$$

$$= \theta_j + \dot{\theta}_{jk}\left(t - t_{\theta_j}\right) + \frac{1}{2}\ddot{\theta}_k\, (t - t_{\theta_k} + \frac{1}{2}t_k)^2$$

$$\dot{\theta}(t) = \dot{\theta}_{jk} + \ddot{\theta}_k\, (t - t_{\theta_k} + \frac{1}{2}t_k) \qquad \ddot{\theta}(t) = \ddot{\theta}_k$$

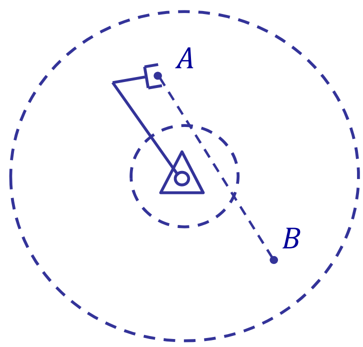
## Linear Function with Parabolic Blends -12

- $\ddot{\theta}$ may not be achievable in actual physical device, determined by
  - Motor specification ($\dot{\theta}$ vs $\tau$ curve)
  - Configuration of the manipulator $f(\theta)$
  - Dynamics of the manipulator

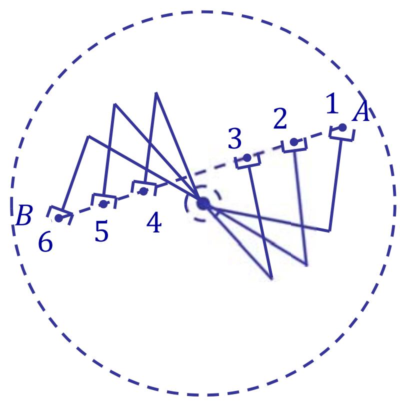


## Geometrical Problems with Cartesian Paths -1

- Unreachable intermediate points

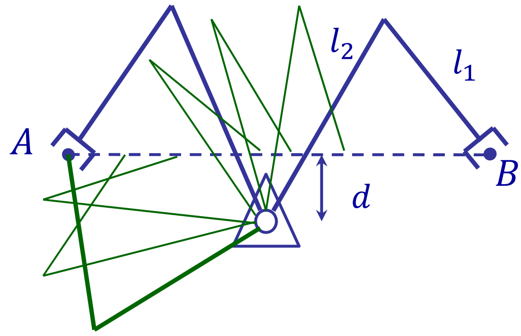



- High joint rates near singularity



## Geometrical Problems with Cartesian Paths -2

- Unreachable goal from given start point & path
  - Unless $l_1 - l_2 = d$, “can flip”






## The End

- Questions?