Chap 8 Frequency Response Method

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Introduction -1

- Performance: in time-domain
  - Transient response (Ex: Rise time T_r, P.O. & peak time T_p, Settling time T_s)
  - Steady-state response (Ex: steady-state error e_{ss})
- Design: in s-domain
  - Routh Hurwitz Criterion
  - Root locus
- Alternative design: in ω–domain
  - Bode plot
  - Nyquist Criterion

Introduction -2

- Frequency Response: The steady-state response of the system to a sinusoidal input

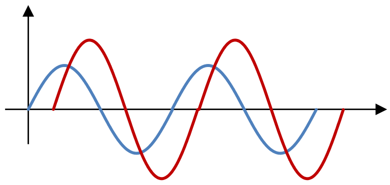
A well-studied periodic signal readily available in many instruments

input → (stable) Linear system, $T(s)$ → output

$r(t) = A\sin(\omega t)$

$y(t) = A|T(j\omega)|\sin(\omega t + \phi)$

$\phi = \angle T(j\omega)$



Input vs. output
Same frequency
Different amplitude & phase angle

Introduction -3

- Derivation

$R(s) = A\frac{\omega}{s^2+\omega^2} \quad T(s) = \frac{p(s)}{\prod_{j=1}^{n}(s+p_j)}$

Assuming distinct poles,
stable system (poles in LHP)

$$Y(s) = T(s)R(s) = \frac{p(s)}{\prod_{j=1}^{n}(s+p_j)}\frac{A\omega}{s^2+\omega^2}$$

$$= \frac{k_1}{s+p_j} + \cdots + \frac{k_n}{s+p_n} + \frac{\alpha_0}{s+j\omega} + \frac{\alpha_0^*}{s-j\omega}$$

$\mathcal{L}^{-1}$

$\alpha_0 \triangleq a + jb \qquad |\alpha_0| = |T(j\omega)|\frac{A}{2} = \sqrt{a^2+b^2}$

$= Y(s)(s+j\omega)|_{s=-j\omega} = T(s)\frac{A\omega}{s-j\omega}|_{s=-j\omega} = T(-j\omega)\frac{A}{2}j$

$= (c-jd)\frac{A}{2}j = \frac{A}{2}(d+jc)$ where $T(j\omega) \triangleq c + jd$

Introduction -4

$$y(t) = \underline{k_1 e^{-p_1 t} + \cdots + k_n e^{-p_n t}} \rightarrow 0 \text{ when } t \rightarrow \infty$$
$$+(a+jb)e^{-j\omega t} + (a-jb)\,e^{j\omega t}$$

$$= \left[e^{j\omega t} + e^{-j\omega t}\right] - jb\left[e^{j\omega t} - e^{-j\omega t}\right]$$
$$= 2a\cos(\omega t) + 2b\sin(\omega t)$$
$$= 2\sqrt{a^2+b^2}\left[\sin(\omega t)\frac{b}{\sqrt{a^2+b^2}} + \cos(\omega t)\frac{a}{\sqrt{a^2+b^2}}\right]$$
$$\triangleq \cos\phi \qquad \triangleq \sin\phi$$
$$= 2|\alpha_0|\sin(\omega t+\phi)$$
$$= A|T(j\omega)|\sin(\omega t+\phi)$$

$$\phi = \tan^{-1}\frac{a}{b} = \tan^{-1}\frac{Re(\alpha_0)}{Im(\alpha_0)} = \tan^{-1}\frac{\frac{A}{2}d}{\frac{A}{2}c} = \tan^{-1}\frac{d}{c} = \angle T(j\omega)$$

Introduction -5

- How to obtain $T(j\omega)$?
  - $\Rightarrow T(j\omega) = T(s)|_{s=j\omega}$ Why?
- Laplace Transform vs. Fourier Transform

Laplace transform

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$$
$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds$$

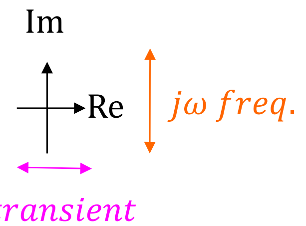
Intergral along vertical lines,
$\sigma > Re(poles)$ to ensure convergence
Here T(s) is stable $\Rightarrow$ Choose $\sigma = 0$

Fourier transform

$$F(\omega) = \mathcal{F}[f(t)] = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$
$$f(t) = \mathcal{F}^{-1}[F(\omega)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

$f(t)$ is defined only in t > 0
$\Rightarrow$ Change integration from $-\infty$ to 0

$s = \sigma + j\omega$

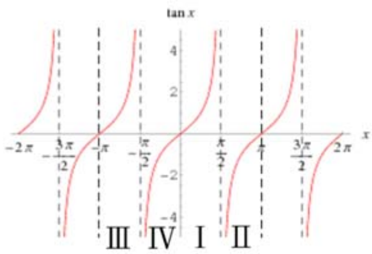


$$interchangable \Rightarrow s = j\omega$$

Polar Plots -1

- Plot magnitude and phase angle of $T(j\omega)$ in complex-plane with varying ω

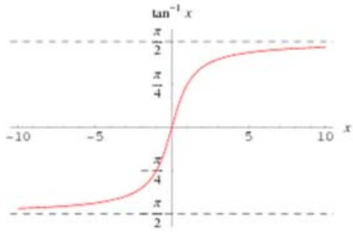
$$G(j\omega)=G(s)|_{s=j\omega}$$
$$= Re[G(j\omega)] + jIm[G(j\omega)]$$
$$= R(\omega) + jX(\omega)$$
$$= |G(\omega)|e^{j\phi(\omega)}$$
$$= |G(\omega)|\angle\phi(\omega)$$



$$|G(j\omega)|^2 = [R(\omega)]^2+[X(\omega)]^2$$

~~$\phi(\omega) = tan^{-1}\frac{X(\omega)}{R(\omega)}$~~ $\quad -\frac{\pi}{2}\sim\frac{\pi}{2}$

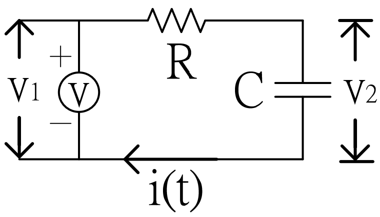
$$= tan2^{-1}(X(\omega), R(\omega)) \quad -\pi\sim\pi$$



Polar Plots -2

- Ex: RC circuit

$$V_R = RI$$
$$V_C = \frac{I}{Cs}$$



$$G(s) = \frac{V_C}{V_R+V_C} = \frac{\frac{I}{Cs}}{RI+\frac{I}{Cs}} = \frac{1}{RCs+1}$$

$$G(j\omega) = \frac{1}{j\omega(RC)+1} = \frac{1}{j(\omega/\omega_1)+1} = \frac{1-j(\omega/\omega_1)}{(\omega/\omega_1)^2+1} \quad \tau = \text{RC},\ \omega_1=\frac{1}{RC}$$

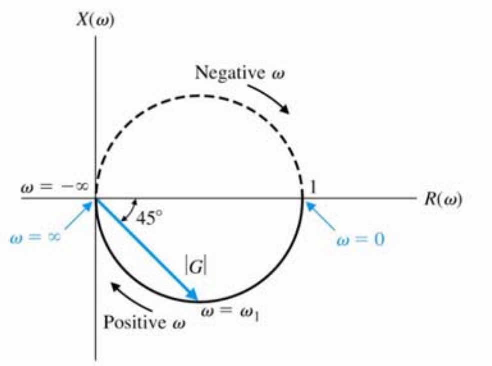
$$= \frac{1}{(\omega/\omega_1)^2+1} + j\frac{-(\omega/\omega_1)}{(\omega/\omega_1)^2+1}$$

a circle centered at $(\frac{1}{2}, 0)$

$$|G(\omega)| = \frac{1}{\sqrt{1+(\frac{\omega}{\omega_1})^2}} \quad \phi(\omega) = tan2^{-1}(-\frac{\omega}{\omega_1}, 1)$$

Trajectory in 4th quadrant

	$R(\omega)$	$X(\omega)$	$\vert G(\omega)\vert$	$\phi(\omega) =$
$\omega = 0$	1	0	1	0
$\omega = \omega_1$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$-45°$
$\omega \to \infty$	0	0	0	$-90°$



Polar Plots -3

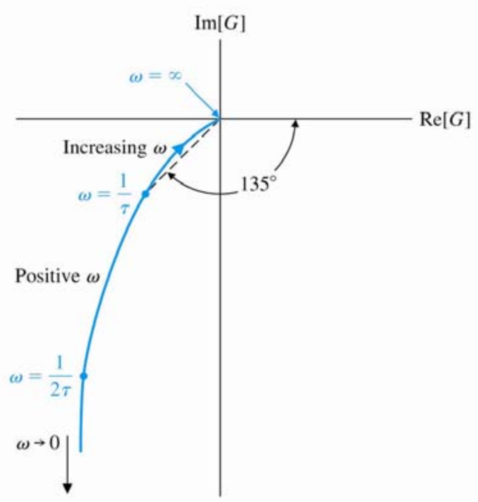
- Ex: $G(s) = \dfrac{k}{s(s\tau + 1)}$ $\quad G(s) = \dfrac{{\omega_n}^2}{s(s + 2\zeta\omega_n)}$ Standard 2nd-order open-loop system

$$G(j\omega) = \frac{k}{j\omega(j\omega\tau+1)} = \frac{-\omega^2 k\tau}{\omega^4\tau^2+\omega^2} + j\frac{-\omega k}{\omega^4\tau^2+\omega^2}$$

$$|G(\omega)| = \frac{k}{(\omega^4\tau^2+\omega^2)^{\frac{1}{2}}}$$

$$\phi(\omega) = tan2^{-1}(-\omega k, -\omega^2 k\tau)$$

$$= tan^{-1}\left(\frac{1}{\omega\tau}\right)$$ Yielding wrong results



	$R(\omega)$	$X(\omega)$	$\|G(\omega)\|$	$\phi(\omega) =$
$\omega = 0$	$-k\tau$	$-\infty$	∞	$-90°$ $90°$
$\omega = \frac{1}{\tau}$	$-\frac{k\tau}{\sqrt{2}}$	$-\frac{k\tau}{\sqrt{2}}$	$\frac{k\tau}{\sqrt{2}}$	$-135°$ $45°$
$\omega \to \infty$	0	0	0	$-180°$ $0°$

Polar Plots -4

- Comments
  - Pros: Polar plot is very useful for investigating system stability (i.e., Nyquist Criterion)
  - Cons:
    - The addition of poles and zeros to an existing system requires the recalculation of the frequency response $\dfrac{k}{s\tau+1}$ vs. $\dfrac{k}{s(s\tau+1)}$
    - The frequency response doesn't indicate the effect of the individual poles or zeros

Bode Plots -1

- Comments
  - Logarithmic plot (or Bode plot) simplifies the determination of the graphical portrayal

$$G(j\omega) = |G(\omega)|e^{j\phi(\omega)}$$

$$\text{Logarithmic gain} = 20log_{10}|G(\omega)|$$ unit: decibels (dB)

⇨ Conversion of multiplicative factors into additive factors

Bode Plots -2

- Ex: Revisit RC circuit

$$G(j\omega)=G(s)\Big|_{s=j\omega} = \frac{1}{RCs+1}\Big|_{s=j\omega} = \frac{1}{j\omega\tau+1} \quad \tau = RC$$

$$G_{dB}(\omega) = 20log|G(\omega)| = 20log(\frac{1}{1+(\omega\tau)^2})^{\frac{1}{2}}$$

$$= -10log(1+(\omega\tau)^2)$$

$$\omega \ll \frac{1}{\tau} \quad G_{dB}(\omega) = -10log(1) = 0\ dB$$

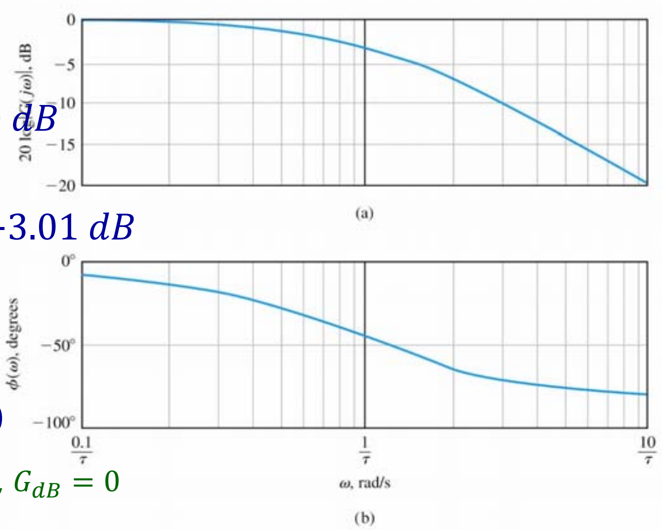
$$\omega = \frac{1}{\tau} \quad G_{dB}(\omega) = -10log(2) = -3.01\ dB$$

$$\omega \gg \frac{1}{\tau} \quad G_{dB}(\omega) = -20log(\omega\tau)$$

$$= -20log(\tau) - 20log(\omega)$$

When $\omega = \frac{1}{\tau}$, $G_{dB} = 0$

$$\phi(\omega) = -tan^{-1}\omega\tau$$



Bode Plots -3

- Comments
  - Decade: An interval of two frequencies with a ratio equal to 10
  - Ex: $G(j\omega) = \dfrac{1}{j\omega\tau + 1}$

Assuming $\omega \gg \frac{1}{\tau}$ $(G_{dB}(\omega) = -20log(\omega\tau))$ and $\omega_2 = 10\omega_1$

$$G_{dB}(\omega_1) - G_{dB}(\omega_2)$$
$$= -20log(\omega_1\tau) + 20log(\omega_2\tau)$$
$$= -20log\frac{\omega_1\tau}{\omega_2\tau}$$
$$= -20log\frac{1}{10}$$
$$= +20\ dB$$

Red lines: Asymptotes

The asymptotic line for this first-order T.F. is -20dB/decade

Bode Plots -4

- Generalized transfer function

$$G(j\omega) = \frac{k_b \prod_{i=1}^{\theta}(1+j\omega\tau_i)}{(j\omega)^N \prod_{m=1}^{M}(1+j\omega\tau_m)\prod_{k=1}^{R}\left[1+\left(\frac{2\zeta_k}{\omega_{nk}}\right)j\omega+\left(\frac{j\omega}{\omega_{nk}}\right)^2\right]}$$ poles in LHP

$$G_{dB} = 20log|G(\omega)| = 20log|\,k_b| + 20\sum_{i=1}^{\theta} log|1 + j\omega\tau_i|$$
$$-20log|(j\omega)^N| - 20\sum_{m=1}^{M} log|1 + j\omega\tau_m|$$
$$-20\sum_{k=1}^{R} log\left|1 + \left(\frac{2\zeta_k}{\omega_{nk}}\right)j\omega + \left(\frac{j\omega}{\omega_{nk}}\right)^2\right|$$

⇨ Bode diagram: Summing the amplitude due to each individual factor

$$\phi(\omega) = \angle k_b + \sum_{i=1}^{Q} tan^{-1}\,\omega\tau_i - N(90°) - \sum_{m=1}^{M} tan^{-1}\,\omega\tau_m$$
$$-\sum_{k=1}^{R} tan^{-1}\left(\frac{2\zeta_k\omega_{nk}\omega}{\omega_{nk}^2-\omega^2}\right)$$

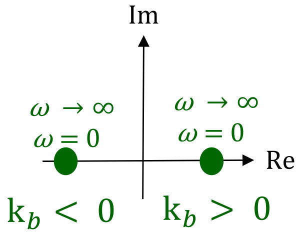
⇨ Phase diagram: Summing the phase angles due to each individual factor

Four Factors -1

- Constant gain, k_b

$$G_{dB} = 20\log|k_b| = constant\ dB$$

$$\phi\,(\omega) = \begin{cases} 0^\circ & k_b > 0 \\ 180^\circ & k_b < 0 \end{cases}$$



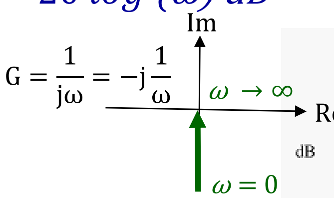
Four Factors -2

- Poles (or zeros) at the origin, $j\omega$

One pole:

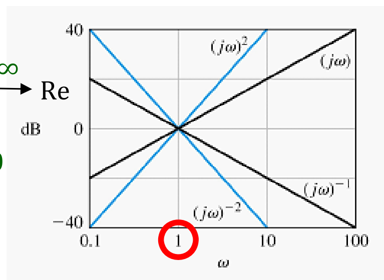
$$G_{dB} = 20log\left|\frac{1}{j\omega}\right| = 20log\frac{1}{\omega} = -20\ log\ (\omega)\ dB$$

$$\phi\,(\omega) = -90°$$



assuming $\omega_2 = 10\omega_1$

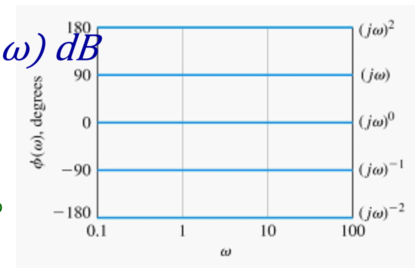
$$20\log\left|\frac{1}{j\omega_1}\right| - 20\log\left|\frac{1}{j\omega_2}\right| = -20(\log\omega_1 - \log\omega_2) = 20\ \text{dB}$$



N poles:

$$G_{dB} = 20log\left|\frac{1}{(j\omega)^N}\right| = 20log\frac{1}{\omega^N} = -20N\ log(\omega)\ dB$$

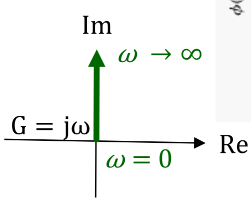
$$\phi\,(\omega) = -90°N$$



One zero:

$$G_{dB} = 20log|j\omega| = 20log(\omega)\ dB$$

$$\phi\,(\omega) = 90°$$



Four Factors -3

- Poles (or zeros) on the real axis, $1 + j\omega\tau$

Pole (LHP):

$$G_{dB} = 20log\left|\frac{1}{1+j\omega\tau}\right| = 20log\frac{1}{\sqrt{1^2+\omega^2\tau^2}} = -10log(1+\omega^2\tau^2)$$

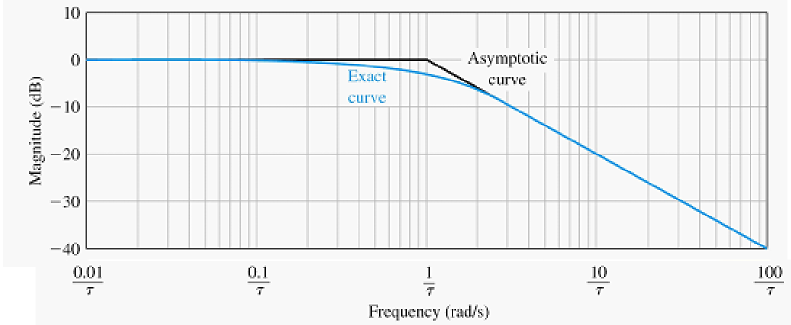
$$\omega \ll \frac{1}{\tau} \quad G_{dB} = -10log(1) = 0\ dB$$

Slope

$$\omega \gg \frac{1}{\tau} \quad G_{dB} = -20log(\omega\tau) = -20log(\tau) - 20log(\omega)\ dB$$

Two asymptotes inter sect at $\omega = \frac{1}{\tau}$, break (or cutoff) frequency

$$\omega = \frac{1}{\tau} \quad G_{dB} = -10log(2) = -3.01\ dB$$



Four Factors -4

Real:

$$\phi\,(\omega) = -\tan^{-1}\omega\tau$$

$$\omega \ll \frac{1}{\tau} \quad \phi = 0$$

$$\omega = \frac{1}{\tau} \quad \phi = -\tan^{-1}1 = -45°$$

$$\omega \gg \frac{1}{\tau} \quad \phi = -90°$$

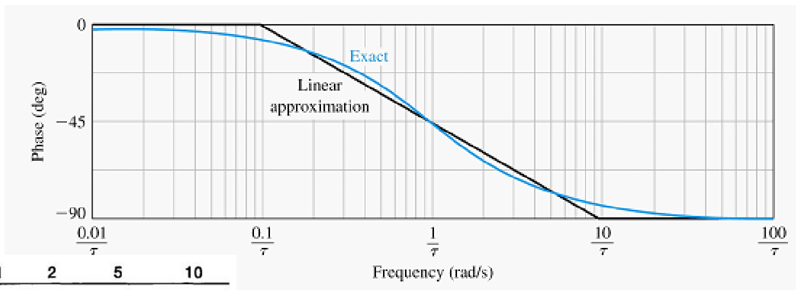
Approximation:

$$\phi\,(\omega) = -90\frac{\log\omega - \log\frac{0.1}{\tau}}{\log\frac{10}{\tau} - \log\frac{0.1}{\tau}} = -45\log\frac{\omega}{\frac{0.1}{\tau}}$$

$$\omega = \frac{0.1}{\tau} \quad \phi = 0$$

$$\omega = \frac{1}{\tau} \quad \phi = -\tan^{-1}1 = -45°$$

$$\omega = \frac{10}{\tau} \quad \phi = -90°$$



$\omega\tau$	0.10	0.50	0.76	1	1.31	2	5	10
$20\log\|(1+j\omega\tau)^{-1}\|$, dB	−0.04	−1.0	−2.0	−3.0	−4.3	−7.0	−14.2	−20.04
Asymptotic approximation, dB	0	0	0	0	−2.3	−6.0	−14.0	−20.0
$\phi(\omega)$, degrees	−5.7	−26.6	−37.4	−45.0	−52.7	−63.4	−78.7	−84.3
Linear approximation, degrees	0	−31.50	−39.5	−45.0	−50.3	−58.5	−76.5	−90.0

Four Factors -5

Zero (LHP):

$$G_{dB} = 20log|1+j\omega\tau| = 10log(1+\omega^2\tau^2)$$

$$\omega \ll \frac{1}{\tau} \quad G_{dB} = 10log(1) = 0\ dB$$

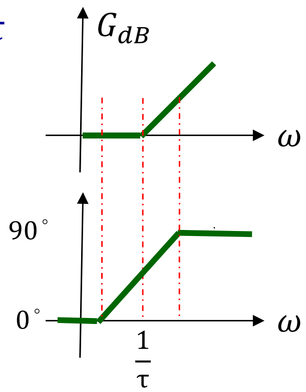
Slope

$$\omega \gg \frac{1}{\tau} \quad G_{dB} = 20log(\omega\tau) = 20log(\tau) + 20log(\omega)\ dB$$

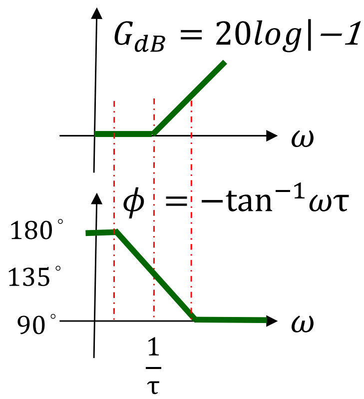
Two asymptotes intersect at $\omega = \frac{1}{\tau}$, break (or cutoff) frequency

$$\omega = \frac{1}{\tau} \quad G_{dB} = 10log(2) = 3.01\ dB$$

$$\phi\ (\omega) = \tan^{-1}\omega\tau$$



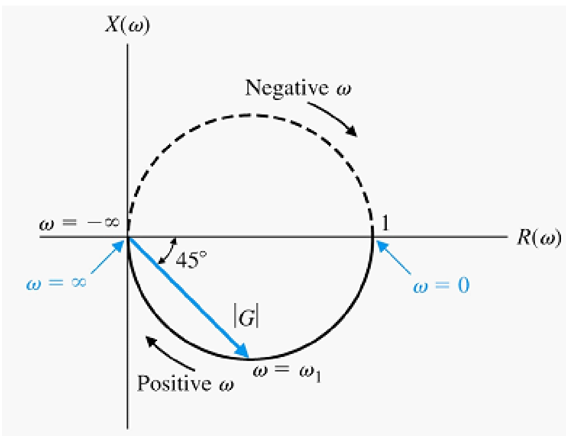
Zero (RHP), $-1 + j\omega\tau$

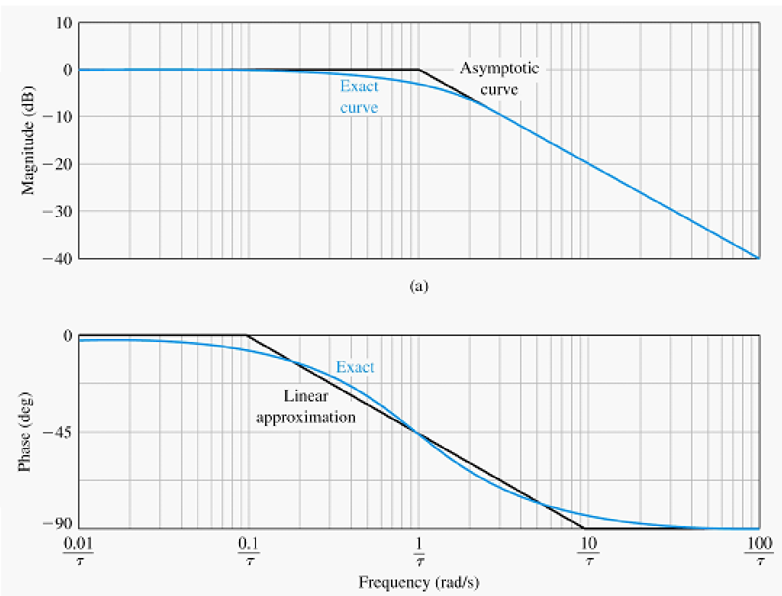


Four Factors -6

- Polar plot vs. bode plot

$$G(j\omega) = \frac{1}{j\omega\tau + 1}$$





Four Factors -7

- Complex conjugate poles or zeros,

$$s^2 + 2\zeta\omega_n s + \omega_n^2 \Rightarrow 1 + \left(\frac{2\zeta}{\omega_n}\right)j\omega + \left(\frac{j\omega}{\omega_n}\right)^2 = 1 + j2\zeta u - u^2 \;,\; u \triangleq \frac{\omega}{\omega_n}$$

$0 \le \zeta \le 1$ Normalize, DC gain=1

Poles:

$$G_{dB} = 20log|G(u)| = 20log\left|\frac{1}{1-u^2+j2\zeta u}\right| = -10log[(1-u^2)^2 + 4\zeta^2u^2]$$

a function of ζ

$$\phi\,(u) = -tan^{-1}\left(\frac{2\zeta u}{1-u^2}\right) \quad or \quad = tan2^{-1}(-2\zeta u, 1-u^2)$$

$u \ll 1 \quad G_{dB} = -10log(1) = 0\;dB$

$\phi\,(u) \approx 0$

Slope

$u \gg 1 \quad G_{dB} = -10log\,u^4 = -\,40log\,u$

$\phi\,(u) \approx -180°$

Two asymptotes intersect at $u = 1$

$u = 1 \quad G_{dB} = -10log(4\zeta^2)$

$\phi = -90°$

Note:

$G_{dB}(\omega_1) - G_{dB}(10\omega_1)$

$= -40\left(log\frac{\omega_1}{\omega_n} - log\frac{10\omega_1}{\omega_n}\right)$

$= +40$

Four Factors -8

$$|G(u)| = \left((1-u^2)^2 + (2\zeta u)^2\right)^{-\frac{1}{2}}$$

$$\frac{dG(u)}{du} = -\frac{1}{2}[(1-u^2)^2 + (2\zeta u)^2]^{-\frac{3}{2}}(4u^3 - 4u + 8u\zeta^2) = 0$$

$(4u^3 - 4u + 8u\zeta^2) = 0$

$\rightarrow \; u = 0, \sqrt{1-2\zeta^2}$

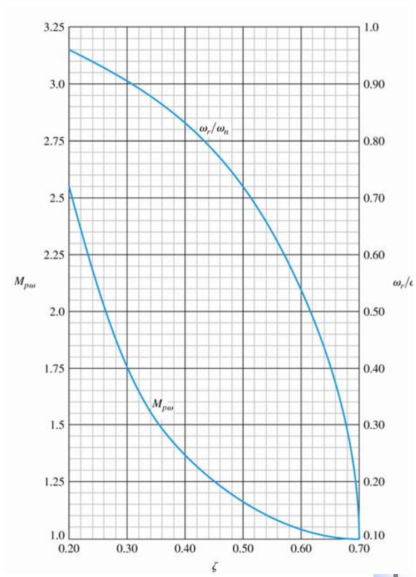
Exist when $0 \le \zeta < \frac{1}{\sqrt{2}}$

$\omega_r = \omega_n u_r = \omega_n\sqrt{1-2\zeta^2}$

Resonant frequency

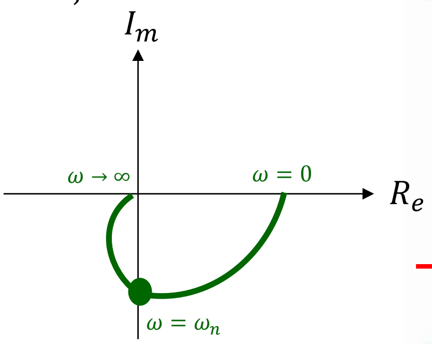
$0 \le \zeta < \frac{1}{\sqrt{2}} \quad M_{p\omega} = |G(u_r)| = \left(2\zeta\sqrt{1-\zeta^2}\right)^{-1}$

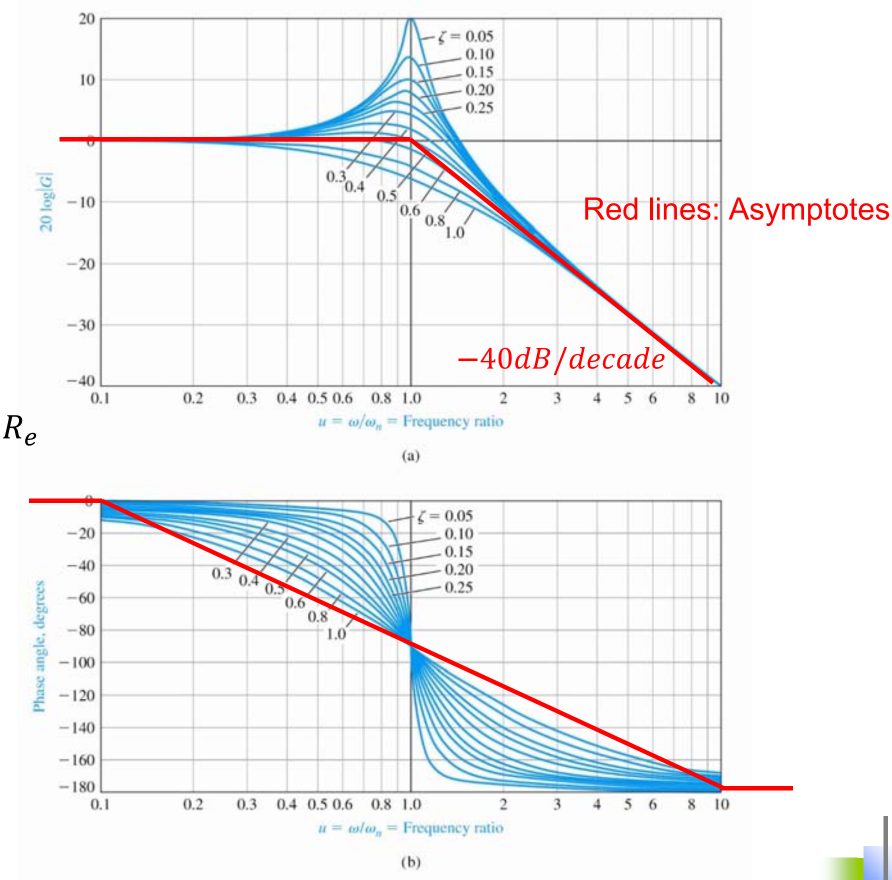
$\frac{1}{\sqrt{2}} \le \zeta \le 1 \quad$ no $M_{p\omega}$ peak



Four Factors -9

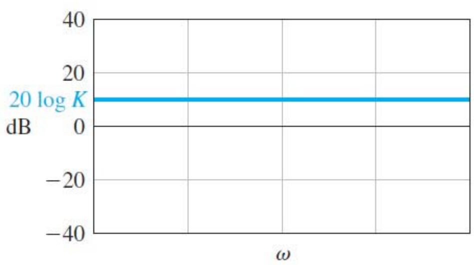
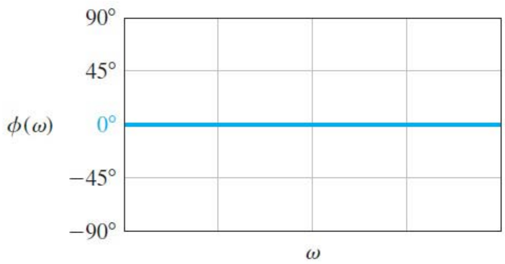
Ex: $\zeta = 0.8$

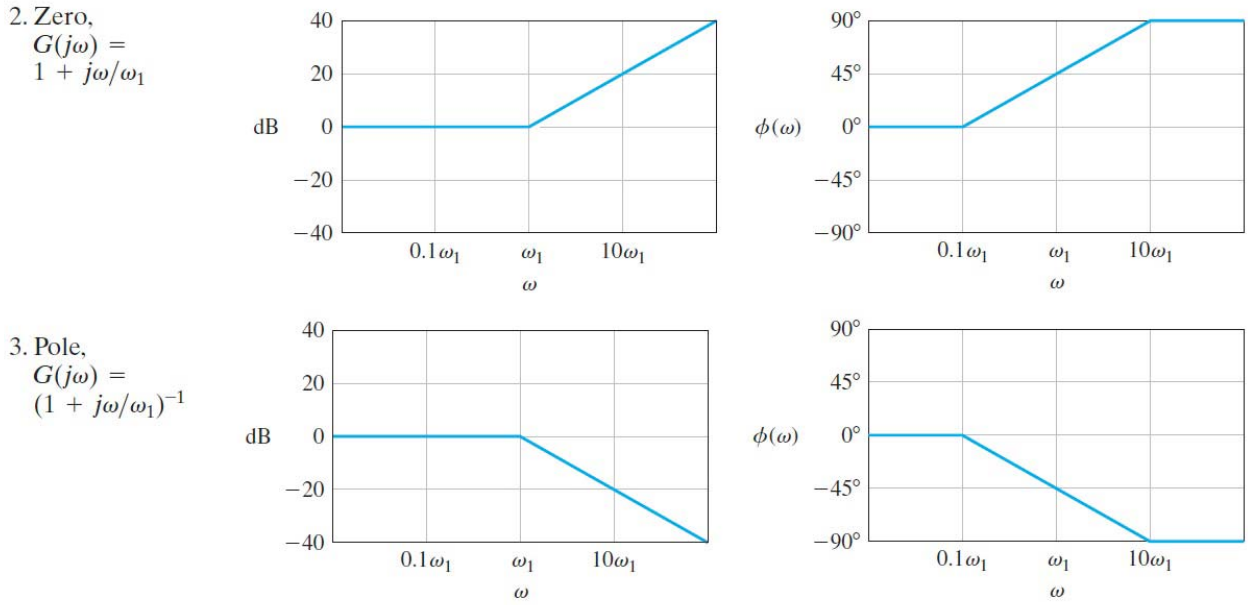




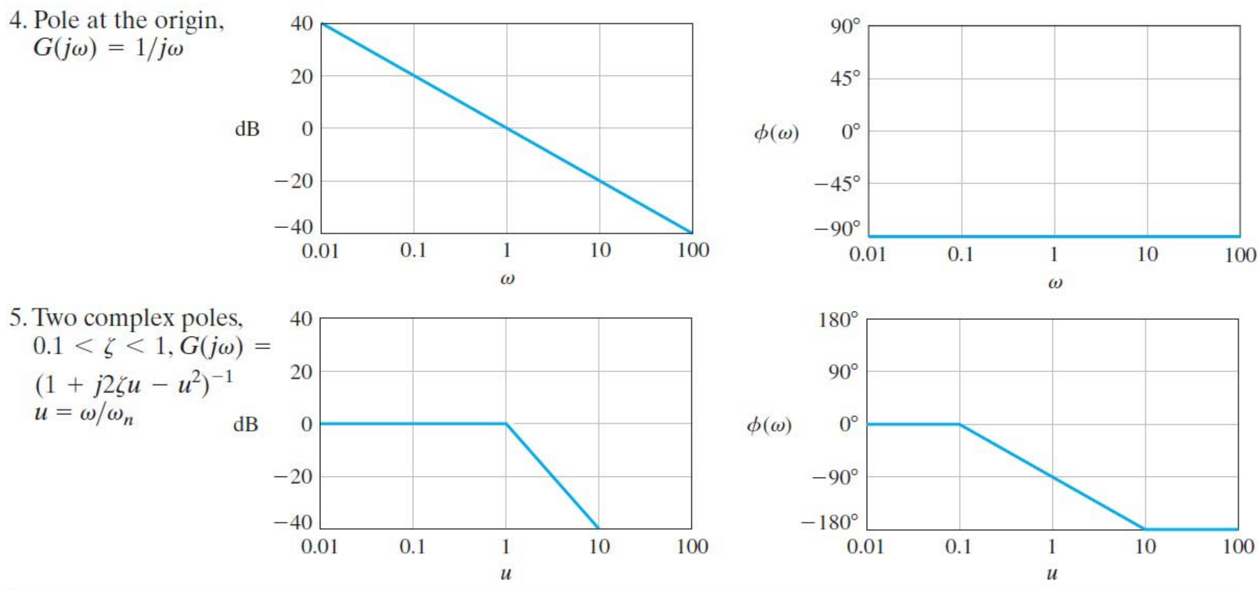
Four Factors -10

Table 8.3 Asymptotic Curves for Basic Terms of a Transfer Function

Term	Magnitude 20 log$\|G\|$	Phase $\phi(\omega)$
1. Gain, $G(j\omega) = K$		



2. Zero,
G(jω) =
1 + jω/ω1
dB
φ(ω)
40
20
0
−20
−40
90°
45°
0°
−45°
−90°
0.1ω1
ω1
10ω1
ω
3. Pole,
G(jω) =
(1 + jω/ω1)^−1



4. Pole at the origin,
G(jω) = 1/jω
dB
φ(ω)
40
20
0
−20
−40
90°
45°
0°
−45°
−90°
0.01
0.1
1
10
100
ω
5. Two complex poles,
0.1 < ζ < 1, G(jω) =
(1 + j2ζu − u²)^−1
u = ω/ωn
180°
90°
0°
−90°
−180°
u

Example -1

- $G(jw) = \dfrac{5\,(1+j0.1w)}{jw(1+j0.5w)\left(1+j0.6\left(\frac{w}{50}\right)+\left(\frac{w}{50}\right)^2\right)}$
  - Step 1: Find asymptotes and adequate w ranges of all factors
    - ① 5, $G_{dB} = 20log5 = 14$, doesn't change with w
    - ② jw, passing $w = 1$ $G_{dB} = 0$, $w = [0.1 \quad 10]$,

      asymptote = $-20logw$
    - ③ $(1 + j0.5w)$, $w_{bf} = 2, w = [0.2 \quad 20]$,

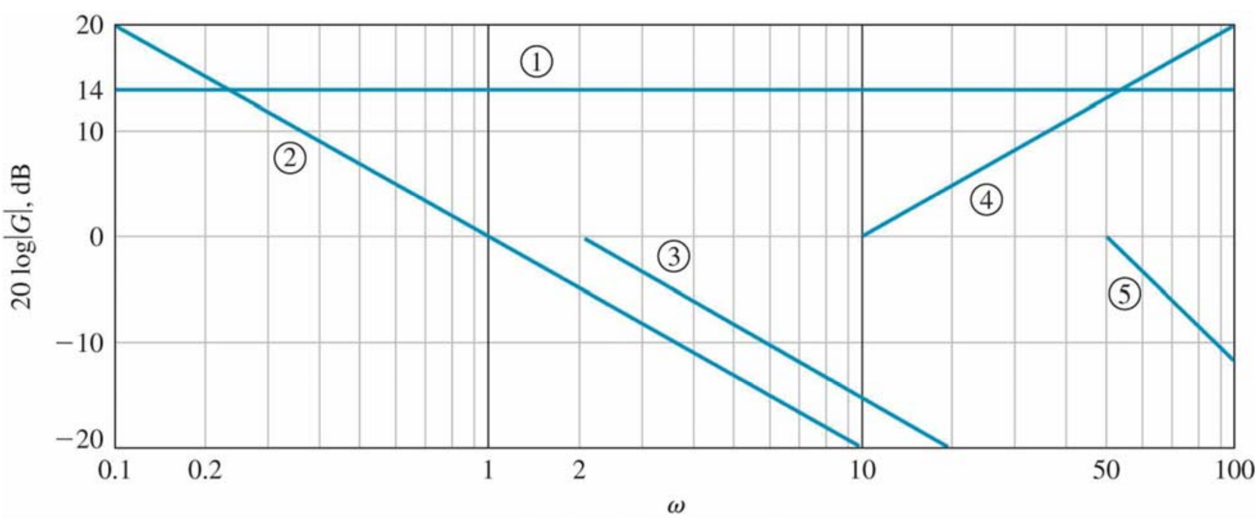
      asymptotes = $\begin{cases} w < 2, G_{dB} = 0 \\ w \geq 2, G_{dB} = 20log2 - 20logw = 6 - 20logw \end{cases}$
    - ④ $(1 + j0.1w)$, $w_{bf} = 10, w = [1 \quad 100]$,

      asymptotes = $\begin{cases} w < 10, G_{dB} = 0 \\ w \geq 10, G_{dB} = 20logw - 20log10 = 20logw - 20 \end{cases}$

Example -2

- ⑤ $\left(1 + j0.6\left(\frac{w}{50}\right) + \left(\frac{w}{50}\right)^2\right)$, $w_{bf} = 50, w = [5 \quad 500]$

  asymptotes = $\begin{cases} w < 50, G_{dB} = 0 \\ w \geq 50, G_{dB} = 40log50 - 40logw = 68 - 40logw \end{cases}$



Example -3

- STEP 2：Decide overall plotting range of $w = [0.1 \quad 500]$ and draw a grid
- STEP 3：Plot the resultant asymptote, starting from low frequency

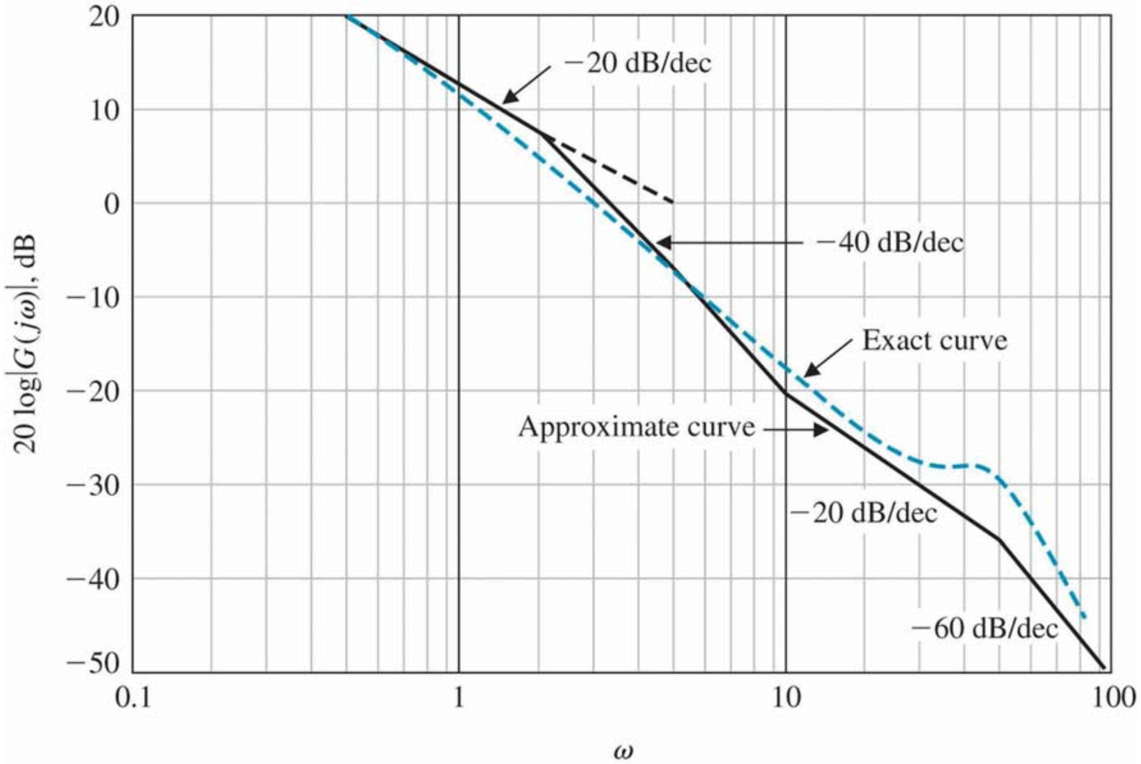
In this example,

  - $0.1 \le w \le 2, G_{dB} = 14 - 20logw$ only ① and ② matter
  - $2 \le w \le 10, G_{dB} = (14 - 20logw) + (6 - 20logw) = 20 - 40logw$ add effect of ③
  - $10 \le w \le 50, G_{dB} = (20 - 40logw) + (20logw - 20) = -20logw$ add effect of ④
  - $w \ge 50, G_{dB} = -20logw + (68 - 40logw) = 68 - 60logw$ add effect of ⑤

P. S. The resultant asymptote should be a continuous piecewise linear function, which means: 將邊界上的 w帶入相鄰的兩線段上，會得到一樣的G_{dB}

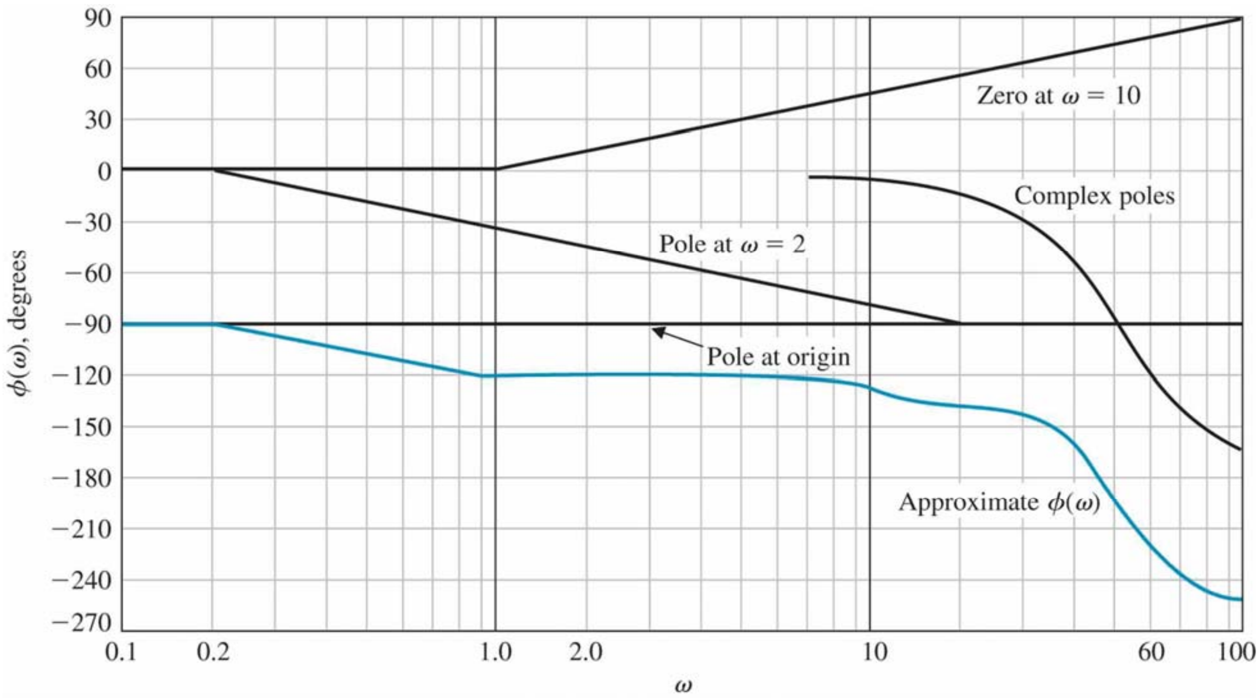
Example -4

- STEP 4：Plot the exact curve



Example -5

- Phase



Get $|G(\omega)|$ graphically

- Ex: $G(s) = \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2} = \dfrac{\omega_n^2}{(s-p_1)(s-p_2)}$

$$p_{1,2} = (-\zeta \pm j\sqrt{1-\zeta^2})\omega_n$$

$$|G(\omega)| = |G(s)||_{s=j\omega} = \frac{\omega_n^2}{|j\omega - p_1||j\omega - p_2|}$$

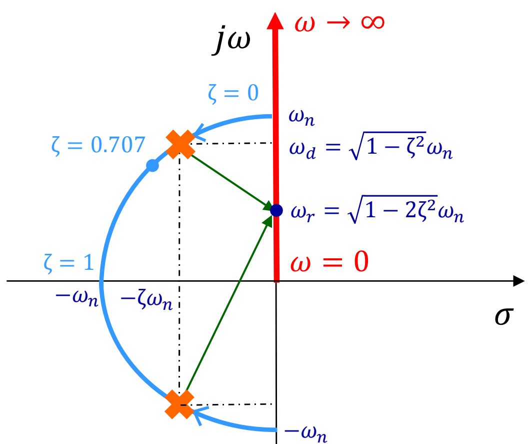
distance from p_1 to $s = j\omega$

$$\phi(\omega) = -\angle(j\omega - p_1) - \angle(j\omega - p_2)$$

$\omega_r = \omega_n\sqrt{1-2\zeta^2}$ $\quad 0 \le \zeta < \frac{1}{\sqrt{2}} = 0.707$

$\Rightarrow s$ 到p_1, p_2兩點距離乘積"最小"

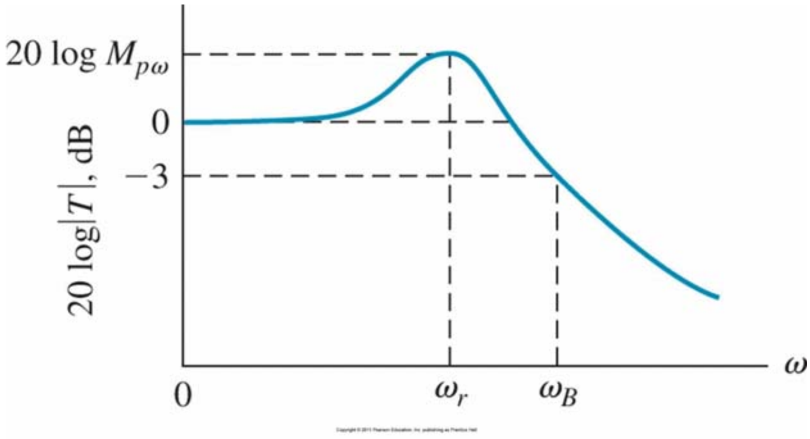
此時，由2個poles和ω_r所形成的
三角形為「直角」三角形



Root locus: varying ζ

Performance Specification -1

- Bandwidth ω_B
  - The frequency at which the frequency response has declined $3\ \mathrm{dB}$ from its low-frequency value



Performance Specification -2

- Ex: Standard 2nd-order system $T(s) = \dfrac{\omega^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
  - Step response

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta) \qquad 0 < \zeta < 1$$

function of ζ only

$$\beta = \sqrt{1-\zeta^2} \qquad \theta = \cos^{-1}\zeta$$

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} \qquad M_{pt} = 1 + e^{-\frac{\zeta\pi}{\sqrt{(1-\zeta^2)}}}$$

Performance Specification -3

- Ex: Standard 2nd-order system $T(s) = \frac{\omega^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$
  - Frequency response, $r(t) = Asin(\omega t)$

$$y(t) = A|G(\omega)|sin(\omega t + \phi(\omega))$$

function of ζ only

$$G_{dB}(u) = -10log[(1-u^2)^2 + 4\zeta^2 u^2]$$

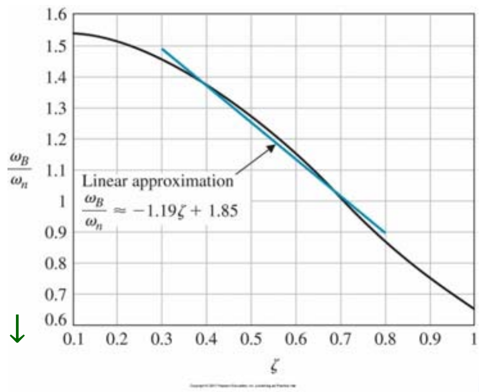
$$\phi\,(u) = tan2^{-1}(-2\zeta u, 1-u^2)$$

$$u \triangleq \frac{\omega}{\omega_n}$$

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \quad M_{pw} = |G(\omega_r)| = \left(2\zeta\sqrt{1-\zeta^2}\right)^{-1} \quad 0 < \zeta < \frac{1}{\sqrt{2}}$$

$$\frac{\omega_B}{\omega_n} = -1.19\zeta + 1.85 \qquad 0.3 \le \zeta \le 0.8$$

$\zeta \uparrow \omega_B \downarrow$



Performance Specification -4

- The magnitude of $M_{p\omega}$ gives indication stability of a stable close-loop system
  - Large $M_{p\omega}$ → Large M_{pt}
- The magnitude of ω_B gives indication of the transient response properties in the time domain
  - Large ω_B → faster t_r
- Desired frequency-domain specifications
  - Relatively small M_{pw}
  - Relatively large bandwidth ω_B

Log Magnitude and Phase Diagrams

- ⇒ logarithmic magnitude (dB) vs. phase angle
- Ex: Revisit

$$G(jw) = \frac{5\,(1+j0.1w)}{jw(1+j0.5w)\left(1+j0.6\left(\frac{w}{50}\right)+\left(\frac{w}{50}\right)^2\right)}$$