Chap 7 The Root Locus Method

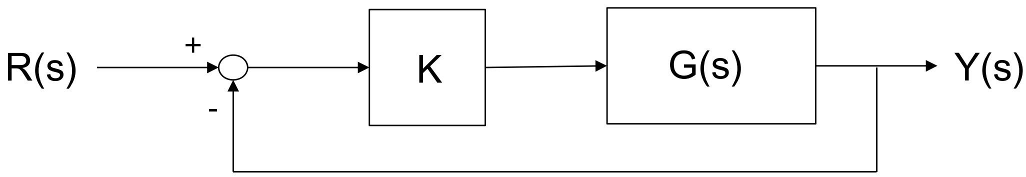
林沛群
國立台灣大學
機械工程學系

Introduction

- Definition
  - The root locus is the path of roots of the characteristic equation traced out in the s-plane as a system parameter is changed
- Motivation
  - It is frequently necessary to adjust one or more system parameters in order to obtain suitable system performance (root locations)

The Root Locus Concept -1

- A simple single loop system



$$T(s) = \frac{Y(s)}{R(s)} = \frac{p(s)}{q(s)} = \frac{KG(s)}{1 + KG(s)}$$

$$K \in [0, \infty) \quad \textit{Root Locus}$$

$$K \in (-\infty, 0\,] \quad \textit{Complimentary Root Locus}$$

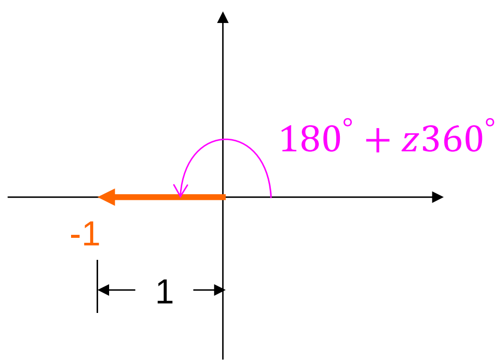
$$K \in (-\infty, \infty) \quad \textit{Complete Root Locus}$$

The Root Locus Concept -2

$$\Delta(s) = 1 + KG(s) = 0$$

Parameter of interest

$$KG(s) = |KG(s)| \angle KG(s) = -1 + j0$$



- Magnitude condition

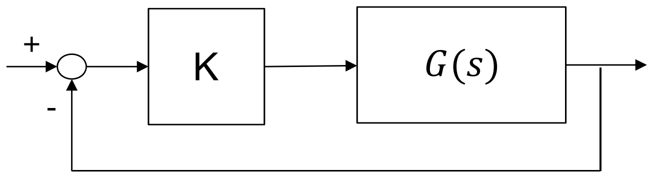
$$|KG(s)| = 1$$

- Angle condition

$$\angle KG(s) = \pi + z(2\pi) = (1 + 2z)\pi \qquad z \in Z$$

The Root Locus Concept -3

- Example : $G(s) = \dfrac{1}{s(s+a)}$ Varying "K"



$$\Delta(s) = 1 + K\frac{1}{s(s+a)} = 0$$

$$s^2 + as + K = 0$$

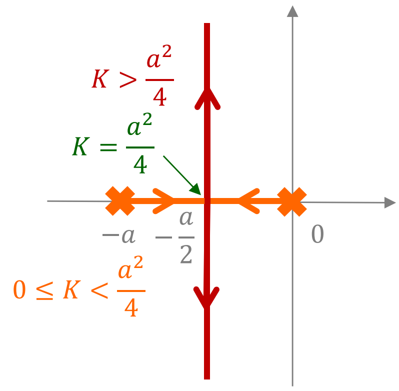
$$s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4K}}{2}$$

- Pole locations

(1) $0 \le K < \dfrac{a^2}{4}$ $\quad s_{1,2} = -\dfrac{a}{2} \pm \dfrac{\sqrt{a^2 - 4K}}{2}$

(2) $K = \dfrac{a^2}{4}$ $\quad s_{1,2} = -\dfrac{a}{2}$

(3) $K > \dfrac{a^2}{4}$ $\quad s_{1,2} = -\dfrac{a}{2} \pm j\dfrac{\sqrt{4K - a^2}}{2}$



The Root Locus Concept -4

- Check with equation constraints

at root s_1

$$\angle\frac{K}{s(s+a)} = 0 - \angle(s_1 - 0) - \angle(s_1 - (-a))$$

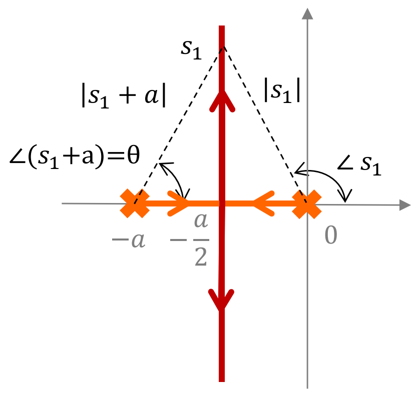
$$= -(\pi - \theta) - \theta = -\pi \quad \text{Satisfy } \pi + z(2\pi) \quad z \in Z$$

$$\left|\frac{K}{s(s+a)}\right|_{s=s_1} = \frac{K}{|s_1||s_1 + a|} = 1$$

$$K = |s_1||s_1 + a|$$

distance from s_1 to -a

distance from s_1 to 0



The Root Locus Concept -5

- Example : $G(s) = \dfrac{1}{s(s+a)}$ Varying "a" (a system parameter)

$$\Delta(s) = 1 + K\frac{1}{s(s+a)} = 0 \qquad s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4K}}{2}$$

- Pole locations

(1) $0 \le a < 2\sqrt{K}$

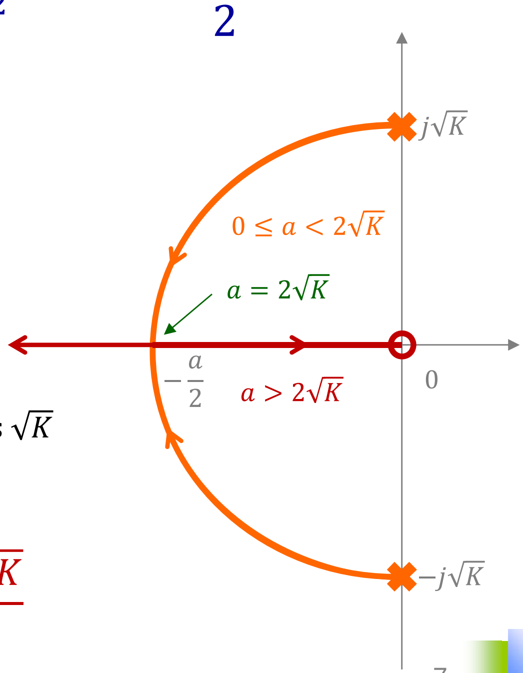
$$s_{1,2} = \frac{-a \pm j\sqrt{4K - a^2}}{2}$$

$$= -\frac{a}{2} \pm j\sqrt{K - \left(\frac{a}{2}\right)^2}$$

A half circle centered at 0 with radius $\sqrt{K}$

(2) $a = 2\sqrt{K}$ $\quad s_{1,2} = -\dfrac{a}{2} = -\sqrt{K}$

(3) $a > 2\sqrt{K}$ $\quad s_{1,2} = -\dfrac{a}{2} \pm \dfrac{\sqrt{a^2 - 4K}}{2}$



The Root Locus Concept -6

- Reformat the system into the standard form

$$\Delta(s) = s^2 + as + K = 0 \qquad 1 + a\frac{s}{s^2 + K} = 0$$

Parameter of interest

- Check with equation constraints :

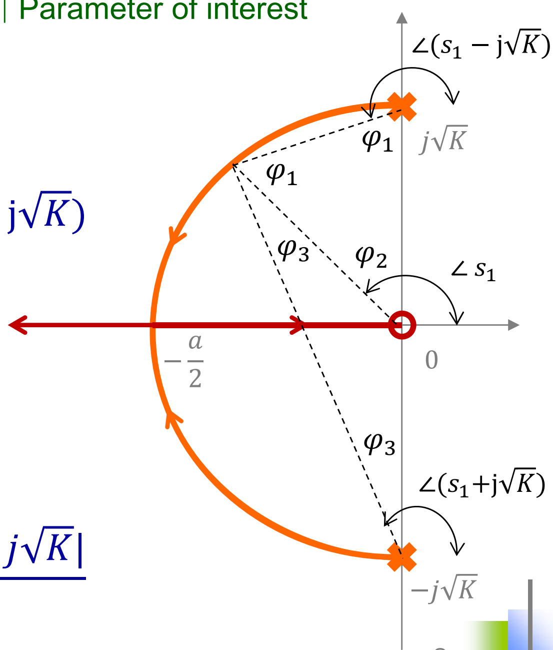
at root s_1

$$\angle\frac{as}{s^2+K} = \angle s_1 - \angle(s_1 + j\sqrt{K}) - \angle(s_1 - j\sqrt{K})$$

$$= \left(\frac{\pi}{2} + \varphi_2\right) - \left(\frac{\pi}{2} + \varphi_3\right) - \left(\frac{3}{2}\pi - \varphi_1\right)$$

$= -\pi$ Satisfy $\pi + z(2\pi)$ $\quad z \in Z$

$$\left|\frac{as}{s^2+K}\right|_{s=s_1} = 1 \qquad a = \frac{|s_1 - j\sqrt{K}||s_1 + j\sqrt{K}|}{|s_1|}$$



The Root Locus Concept -7

- General R.L form

$$\Delta(s) = 1 + F(s) = 0 \qquad F(s) = -1 + j0$$

$$F(s) = \frac{K(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$

Parameter of interest

- Angle condition

$$\angle F(s) = \angle K + (\angle(s+z_1) + \angle(s+z_1) + \dots \angle(s+z_m))$$
$$-(\angle(s+p_1) + \angle(s+p_1) + \cdots \angle(s+p_n))$$
$$= \pi + z(2\pi) \quad z \in Z$$

- Magnitude condition

$$|F(s)| = \frac{K|s+z_1||s+z_2|\dots|s+z_m|}{|s+p_1||s+p_2|\dots|s+p_n|}$$

The Root Locus Procedure -1

- Step 1 : Prepare the root locus sketch
  - (a) Write the characteristic equation such that the parameter of interest, K, appears as a multiplier

$$\Delta(s) = 1 + F(s) = 0 = 1 + KP(s) \quad 0 \le K \le \infty$$

An open-loop T.F. with unity feedback

  - (b) Factor $P(s)$ in terms of n poles and M zeros

$$P(s) = \frac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}$$

The Root Locus Procedure -2

- (c) Locate the open-loop poles and zeros of P(s) in the s-plane

$$\prod_{j=1}^{n}(s+p_j)+K\prod_{i=1}^{M}(s+z_i)=0$$

$K=0$ $\quad \prod_{j=1}^{n}(s+p_j)=0$ R.L. originates from poles of P(s)

$K\to\infty$ $\quad \prod_{i=1}^{M}(s+z_i)=0$ R.L. terminates at zeros of P(s)

- (d) Determine the number of separate loci (SL) = # of poles

  $n-M$ branches of R.L. approached $n-M$ zeros at "infinity"

- (e) The R.L. are symmetrical with repeat to the horizontal real axis

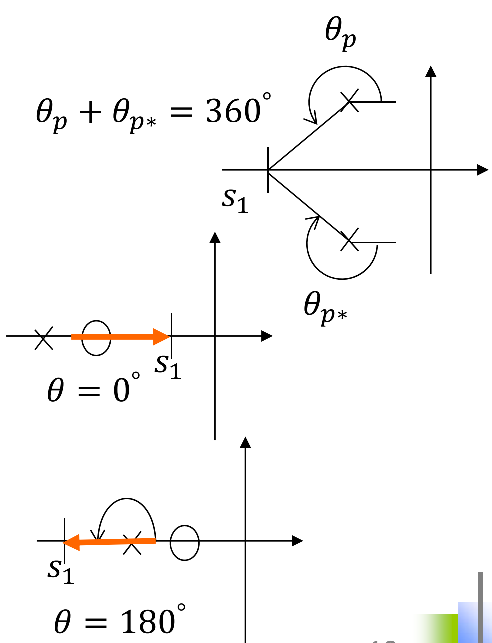
  (i.e., complex conjugate pairs of roots)

The Root Locus Procedure -3

- Step 2 : Locate the segments of the real axis that are root loci

⇨ Always lies in a section of the real axis to the left of an odd number of poles and zeros.

Because :



(1) Complex conjugate poles (or zeros) do not give contribution to the angle condition

(2) Contribution of real poles and zeros to the left of the test points s_1 on the real axis is "0°"

(3) Contribution of each real pole or zero to the right of test point s_1 on the real axis is " 180°"

The Root Locus Procedure -4

- Step 3 : The Loci proceed to the zeros at infinity along asymptotes ($center\ at\ \sigma_A,\ with\ angle\ \varphi_A$)

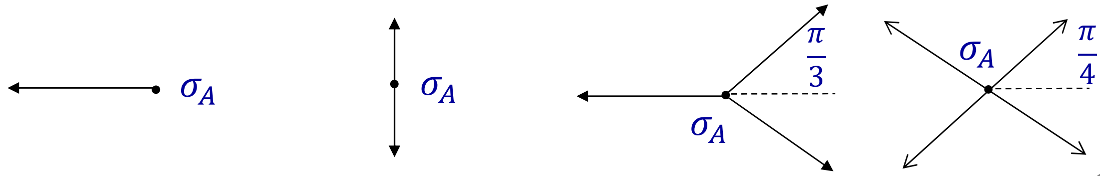
⇨ Use $1 + K\dfrac{1}{(s-\sigma_A)^{n-M}} = 0$ to get approximation (當s大時，低階項量值小可忽略)

$$(s-\sigma_A)^{n-M} = -K$$

$$s = \sigma_A + (-K)^{\frac{1}{n-M}} = \sigma_A + \left[Ke^{i(\pi+2z\pi)}\right]^{\frac{1}{n-M}}$$

$$= \sigma_A + K^{\frac{1}{n-M}} e^{i\left(\frac{\pi}{n-M}+\frac{2\pi}{n-M}z\right)} \qquad z = 0,1,2,\ldots,(n-M-1)$$

$n-M=1$ $\quad$ $n-M=2$ $\quad$ $n-M=3$ $\quad$ $n-M=4$



The Root Locus Procedure -5

- Thus, $\varphi_A = \dfrac{(2z+1)}{n-M}\pi$

$$= \frac{\pi}{n-M} + \frac{2\pi}{n-M}z \qquad z = 0,1,2,\ldots,(n-M-1)$$

$$orig\ sys = 1 + K\frac{\prod^M (s+z_i)}{\prod^n (s+p_j)} = 1 + K\frac{s^M + b_{M-1}s^{M-1} + \cdots}{s^n + a_{n-1}s^{n-1} + \cdots}$$

$$= 1 + K\frac{1}{s^{n-M} + (a_{n-1} - b_{M-1})s^{n-M-1} + \cdots}$$

$$approx.sys = 1 + K\frac{1}{(s-\sigma_A)^{n-M}} \qquad a_{n-1} = \sum_{j=1}^{n} p_j \quad b_{M-1} = \sum_{i=1}^{M} z_i$$

$$= 1 + K\frac{1}{s^{n-M} - (n-M)\sigma_A s^{n-M-1} + \cdots}$$

$$\sigma_A = \frac{a_{n-1} - b_{M-1}}{-(n-M)} = \frac{\sum(-p_j) - \sum(-z_i)}{n-M}$$

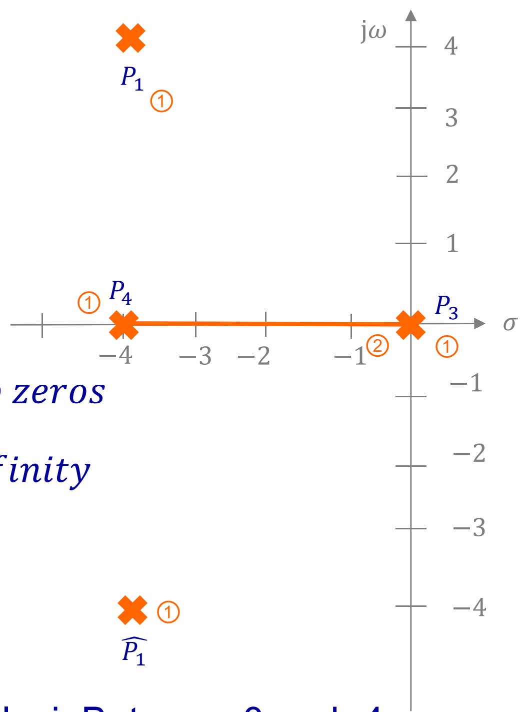
The Root Locus Procedure -6

- Step 4: Determine the points at which the locus crosses the imaginary axis (If it does so)

  ⇨ Use Routh-Hurwitz Criterion

Root Locus Example -1

- $s^4 + 12s^3 + 64s^2 + 128s + K = 0$
- Step 1
  - (a) $1 + K\frac{1}{s^4+12s^3+64s^2+128s} = 0$
  - (b) $1 + K\frac{1}{s(s+4)(s+4+j4)(s+4-j4)} = 0$
  - (c) $4\ open-loop\ poles; no\ open-loop\ zeros$
  - (d) $4\ seperate\ loci;\ 4\ loci\ approach\ infinity$
  - (e) $Symmetry\ w.r.t.the\ real\ axis$



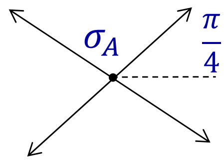
- Step 2
  - Segments of the real axis that are root loci: Between 0 and -4

Root Locus Example -2

- $s^4 + 12s^3 + 64s^2 + 128s + K = 0$
- Step 3: Asymptotes centered at σ_A and with angle φ_A

$$\sigma_A = \frac{(-4) + (-4 + 4j) + (-4 - 4j) - 0}{4} = -3$$

$$\varphi_A = \frac{2z + 1}{4}\pi \bigg|_{z=0,1,2,3} = \frac{1}{4}\pi, \frac{3}{4}\pi, \frac{5}{4}\pi, \frac{7}{4}\pi$$



Root Locus Example -3

- $s^4 + 12s^3 + 64s^2 + 128s + K = 0$
- Step 4: Locus crosses the jw-axis

s^4	1	64	K
s^3	12	128	
s^2	53.33	K	
s^1	$c_1 = \frac{6822 - 12K}{53.33}$	0	
s^0	K		

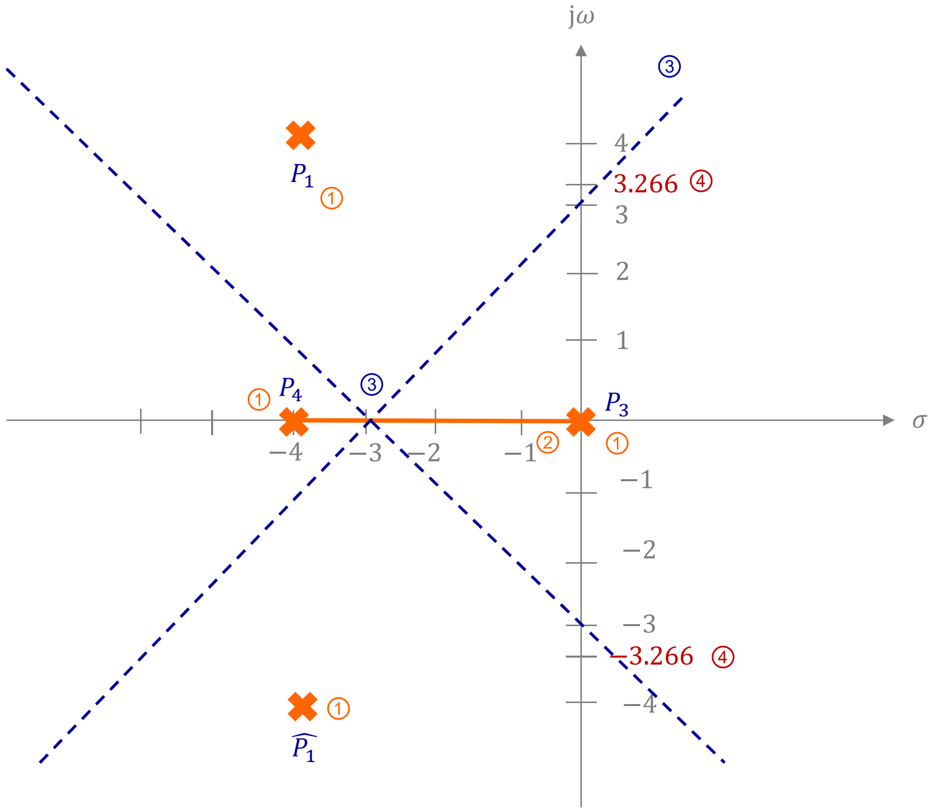
If $c_1 < 0$, two sign changes, 2 roots in RHP

If $c_1 = 0$ (i.e., $K = 568.89$), roots of $U(s) = 0$ are roots of the system

$$U(s) = 53.33s^2 + 568.89 = 53.33(s + j3.266)(s - j3.266) = 0$$

$$s = \pm j3.266$$

Root Locus Example -4



The Root Locus Procedure -7

- <u>Step 5</u>: Determine the breakaway point on the real axis (if any) due to angle criterion

  Corresponding to multiple-order roots of the equation

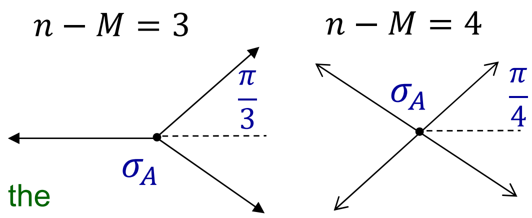
$$1 + F(s) = 1 + KP(s) = 0 \qquad K = -\frac{1}{P(s)}$$

$\Rightarrow$ $\dfrac{dK}{ds} = 0$ to get s

Necessary but NOT sufficient, need to check whether s is on the locus or not

The tangents to the loci at the breakaway points are equally spaced over 360°

$$1 + K\frac{1}{(s-\sigma_A)^{n-M}} = 0$$



Existence of other poles/zeros doesn't affect the angles of departure/arrival of s at the breakaway points

The Root Locus Procedure -8

- Step 6: Determine the angle of departure of the locus from a pole and the angle of arrival of the locus at a zero

⇨ The angle of locus $\begin{cases} arival\ at\ a\ zero \\ departure\ from\ a\ pole \end{cases}$ is the difference between the net angle due to all other poles and zeros and the angle criterion of $\pi(1+2z) \quad z \in Z$

$$\angle(s+p_1) = \angle K + (\angle(s+z_1) + \angle(s+z_1) + \cdots + \angle(s+z_m)) - (\angle(s+p_2) + \cdots + \angle(s+p_n)) - (1+2z)\pi$$

The Root Locus Procedure -9

- Step 7: Complete the sketch
  - Sketching in all sections of the locus NOT covered in the previous steps
  - Use magnitude condition to determine the parameter value K_x at a specific root S_x

$$K_x = \left. \frac{\prod_{j=1}^{n} |s+p_j|}{\prod_{i=1}^{M} |s+z_i|} \right|_{s=s_x}$$

Root Locus Example -5

- $s^4 + 12s^3 + 64s^2 + 128s + K = 0$
- Step 5:

$$K = -s^4 - 12s^3 - 64s^2 - 128s$$

$$\frac{dK}{ds} = 0 = -4(12s^3 + 9s^2 + 32s + 32)$$

$$s = -3.71 \pm j2.55, \underline{-1.58}$$

Choose this one

$$K = 83.57$$

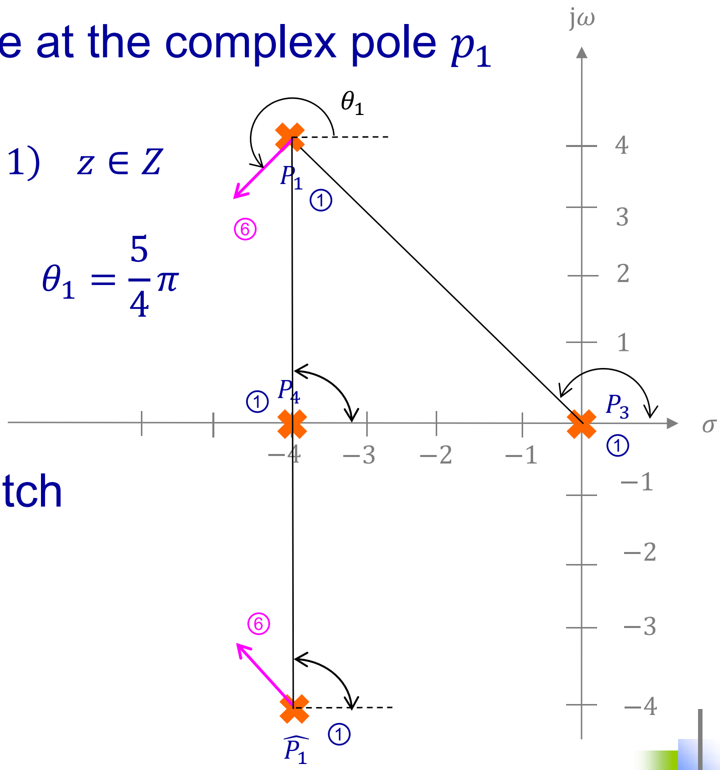
Root Locus Example -6

- $s^4 + 12s^3 + 64s^2 + 128s + K = 0$
- Step 6: Angle of departure at the complex pole p_1

$$-\theta_1 - \frac{\pi}{2} - \frac{\pi}{2} - \frac{3}{4}\pi = \pi(2z+1) \quad z \in Z$$

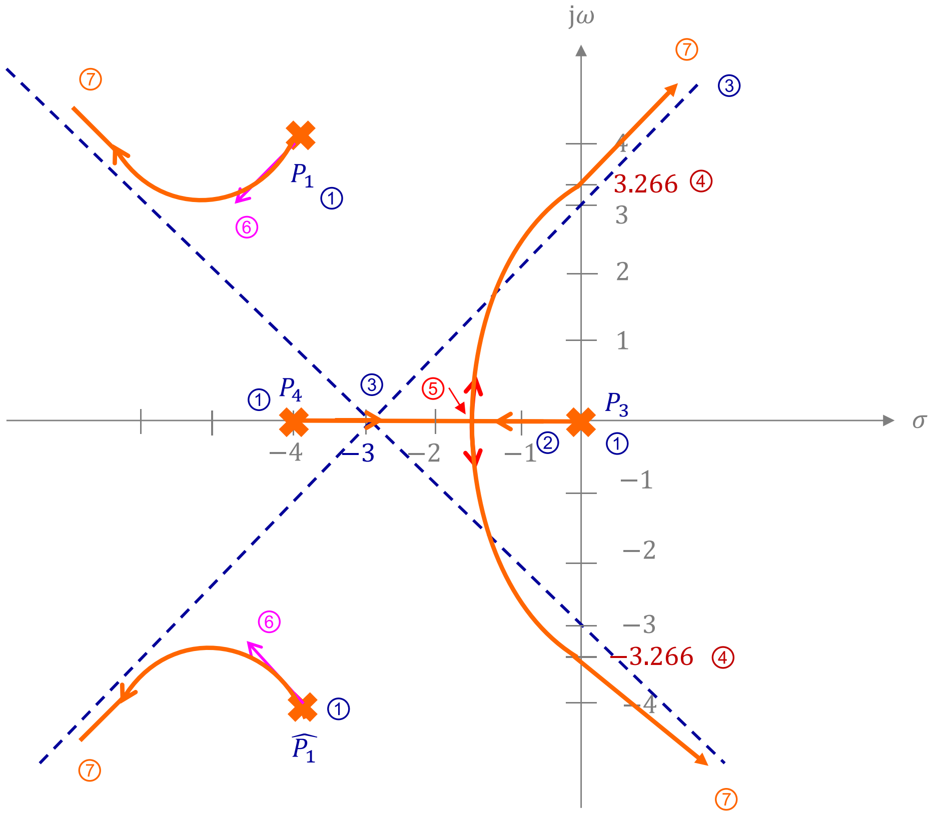
$\uparrow$ $\widehat{p_1}$ $\quad\uparrow$ p_4 $\quad\uparrow$ p_3

$$\theta_1 = \frac{5}{4}\pi$$



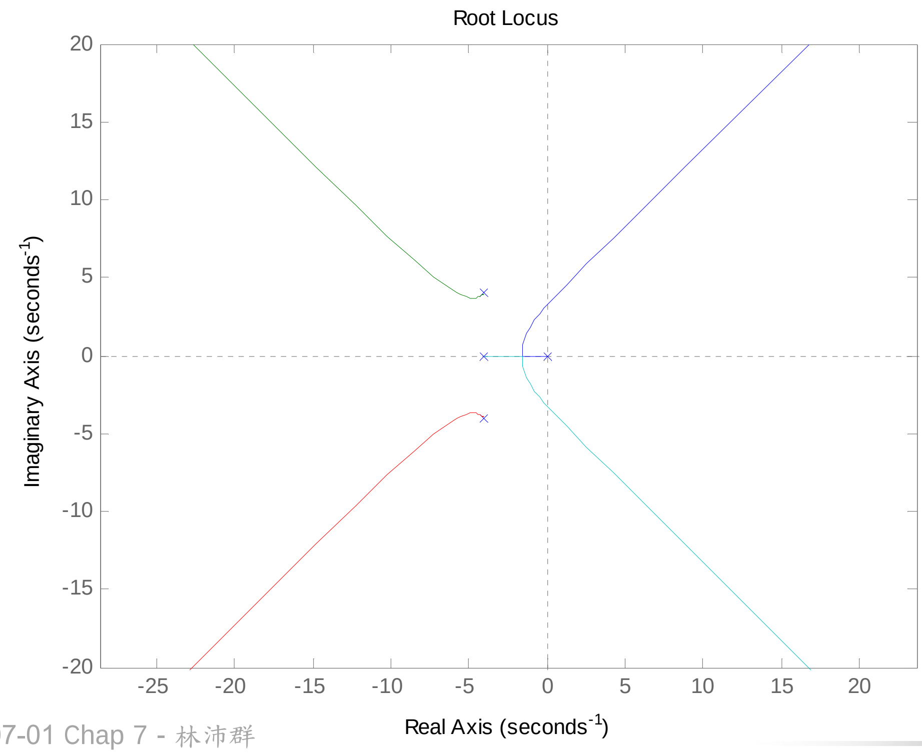
- Step 7: Complete the sketch

Root Locus Example -7



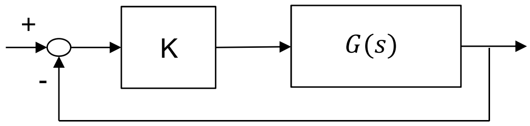
Root Locus Example -8

- Matlab
  - rlocus(tf(1, [1 12 64 128 0])); axis equal;



Parameter Design by the Root Locus Method -1

- Root Locus
  - Not only for determining the R.L. of $\Delta(s)$ as the system gain K changes $(0 \rightarrow \infty)$



  - Can also be used to check effect of the system parameters

    Method: Isolate the parameter and rewrite the equation in standard root locus form

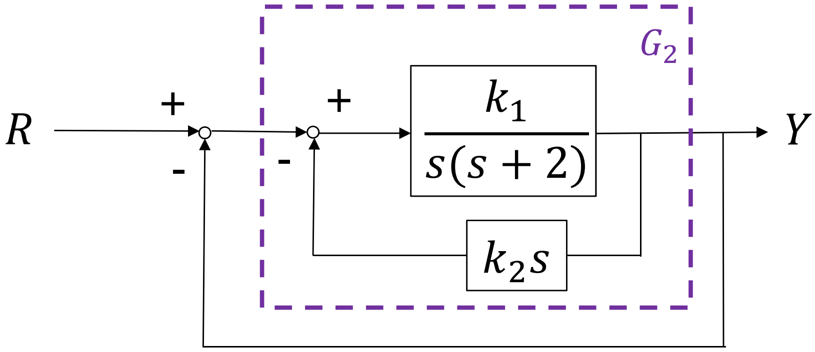
  - Ex: $G = \frac{1}{s(s+a)}$, varying "a" ⇨ $1 + a\frac{s}{s^2 + K} = 0$

Parameter Design by the Root Locus Method -2

- If having two parameters: Two-step method
  - First, evaluating effect of the parameter, α, by R.L. and select a suitable α
  - Second, repeat the above process for the parameter, β, by R.L.
  - Limitation: not always able to obtain a characteristic equation that is linear in the parameter under consideration

Example: Welding Head Control -1

- Block diagram



- Requirements

1) $e_{ss} \le 35\%\ for\ a\ ramp\ input$
2) Damping ratio of dominant roots ≥ 0.707
3) $T_s\ (2\%\ of\ the\ final\ value) \le 3\ sec$

Example: Welding Head Control -2

- Suitable pole locations

$$G_2 = \frac{\frac{k_1}{s(s+2)}}{1+\frac{k_1}{s(s+2)}k_2 s} = \frac{k_1}{s(s+2)+k_1k_2 s}$$

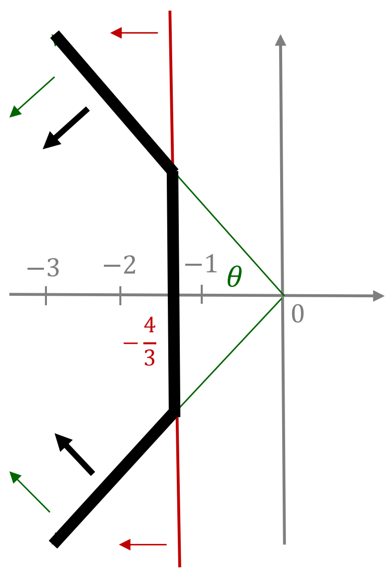
$$E = R - Y = \left(1 - \frac{G_2}{1+G_2}\right)R = \frac{1}{1+G_2}R$$

$$e_{ss} = \lim_{s\to 0} s \frac{1}{1+\frac{k_1}{s(s+2)+k_1k_2 s}} \frac{A}{s^2} = \frac{2+k_1k_2}{k_1}A$$

$$\frac{e_{ss}}{A} = \frac{2+k_1k_2}{k_1} \le 0.35$$

$\zeta \ge 0.707$ $\quad\theta \le 45°$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma} \le 3 \qquad \sigma \ge \frac{4}{3}$$



Example: Welding Head Control -3

- Characteristic equation

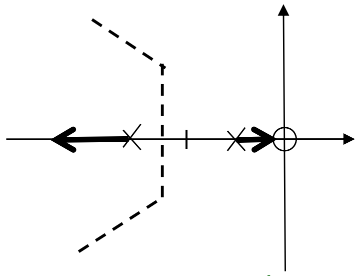
$$\Delta(s) = s(s+2) + k_1 k_2 s + k_1 = 0$$

Assume $\alpha = k_1,\ \ \beta = k_1 k_2$

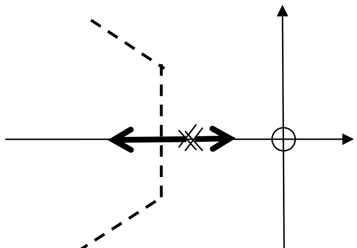
Determine α first, then β

$$\frac{e_{ss}}{A} = \frac{2+\beta}{\alpha}$$

$$1 + \beta \frac{s}{s^2 + 2s + \alpha} = 0$$



$\alpha: 2\ real\ poles - doesn't\ work$



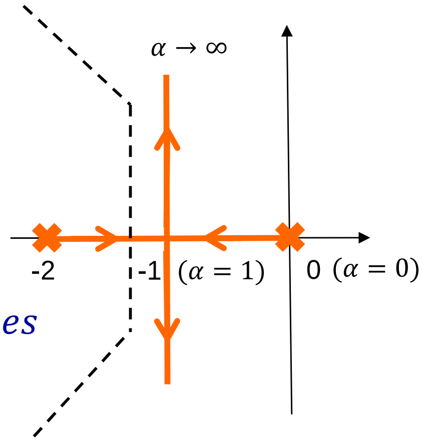
$\alpha: 1\ repeated\ poles\text{-}doesn't\ work$

- Step 1: set $\beta = 0$

$$s^2 + 2s + \alpha = 0$$

$$1 + \alpha \frac{1}{s(s+2)} = 0$$

⇨ $\alpha: 2\ conjugate\ poles$



Example: Welding Head Control -4

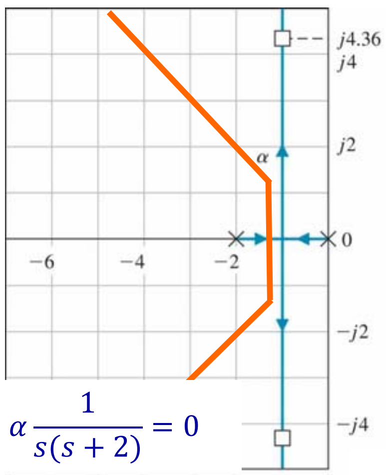
- Step 2:

$$1 + \beta \frac{s}{s^2 + 2s + \alpha} = 0$$

◆ 課本方法

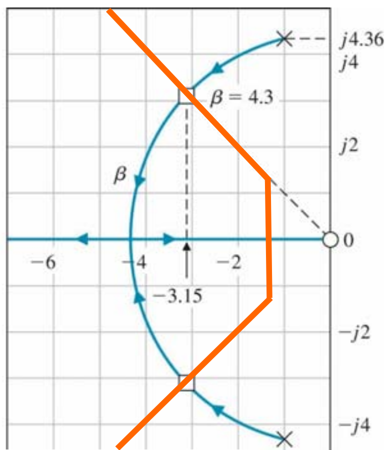
$choose\ \alpha = 20 = k_1$

$\therefore \beta = 20 k_2$



$$1 + \alpha \frac{1}{s(s+2)} = 0$$

(a)



$$1 + \beta \frac{s}{s^2 + 2s + \alpha} = 0$$

$$plot\ 1 + \beta \frac{s}{s^2 + 2s + 20} = 0$$

$choose\ \zeta = 0.707 \quad \therefore \sigma = 3.15 \quad \& \quad \beta = 4.3 \quad \rightarrow \quad k_2 = 0.215$

$\downarrow$

$T_s = 1.27 < 3\ sec \qquad \frac{e_{ss}}{A} = \frac{2 + k_1 k_2}{k_1} = 0.315 \le 0.35$

Example: Welding Head Control -5

- 解析法

  assume $s = -c + jd$ $\begin{cases} c > \dfrac{4}{3} \text{ to meet } T_s \le 3 \\ 0 \le d \le c \text{ to meet } \zeta > 0.707 \end{cases}$

$$\Delta(s) = s^2 + 2s + \beta s + \alpha \Big|_{s=-c+jd}$$

$$= \underbrace{(c^2 - d^2 - 2c - \beta c + \alpha)}_{①} + j\underbrace{(2d - 2cd + \beta d)}_{②}$$

$$= d(2 - 2c + \beta) = 0$$

$$(2 - 2c + \beta) = 0$$

$$\beta = 2c - 2 \ \text{ into } ①$$

$$\alpha = d^2 - c^2 + 2c + \beta c$$

$$= d^2 + c^2$$

$$\Rightarrow \frac{e_{ss}}{A} = \frac{(2+\beta)}{\alpha} = \frac{2c}{d^2 + c^2} < 0.35$$

Example: Welding Head Control -6

$$0.35c^2 - 2c + 0.35d^2 > 0$$

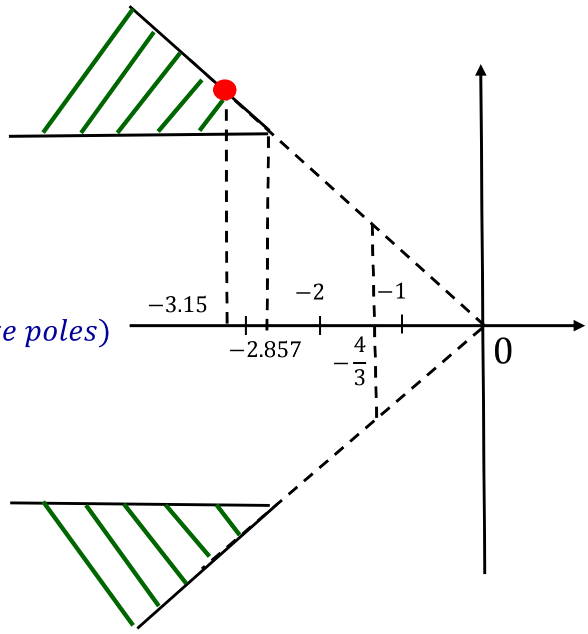
判別式 $= (-2)^2 - 4(0.35)(0.35d^2)$

$$= 4(1 - 0.35^2 d^2) < 0$$

$$\Rightarrow d > \frac{1}{0.35} = 2.857$$

(*i.e.*, 必為 *complex conjugate poles*)

$$c \ge d = 2.857$$



choose root $(-c + jd)$ *within red area*

calculate $\alpha = c^2 + d^2$ *and* $\beta = 2c - 2$

Sensitivity and the Root Locus -1

- How does variation of the system gain, K, (or system parameter, β) alter the root location?

- Ex: $1 + K\frac{1}{s(s+\beta)} = 0$

$$K = K_0 + \Delta K = 0.5 + \Delta K$$
$$\beta = \beta_0 + \Delta\beta = 1 + \Delta\beta$$

Nominal values

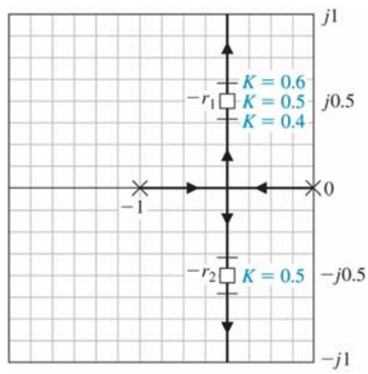
$nominal\ \ operating\ point$:

$$1 + \frac{K}{s(s+1)} = 0 \qquad K_0 = 0.5$$
$$s_{1,2} = -0.5 \pm j0.5$$

Sensitivity and the Root Locus -2

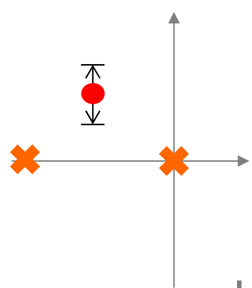
- Variation of K

$$\Delta(s) = s^2 + s + (0.5 + \Delta K) = 0$$
$$1 + \Delta K\frac{1}{s^2 + s + 0.5} = 0$$
$$20\%\ variation \qquad \Delta K = \pm 0.1$$



$\Rightarrow$ $K = 0.6$ $\quad s_{1,2}\Big|_{k=0.6} = -0.5 + j0.59$

$$\Delta s_1 = j0.09$$



$\Rightarrow$ $K = 0.4$ $\quad s_{1,2}\Big|_{k=0.4} = -0.5 - j0.387$

$$\Delta s_1 = -j0.11$$

Sensitivity and the Root Locus -3

- Root Sensitivity ≡ $S_K^{r_i} = \frac{\partial r_i}{\partial K / K} \approx \frac{\Delta r_i}{\Delta K / K}$

$$S_{K+}^{s_1} = \frac{\Delta s_1}{\Delta K/K} = \frac{j0.9}{0.2} = j0.45 = 0.45\angle{+90^\circ}$$

取正值

$$S_{K-}^{s_1} = \frac{\Delta s_1}{\Delta K/K} = \frac{-j0.11}{0.2} = -j0.55 = 0.55\angle{-90^\circ}$$

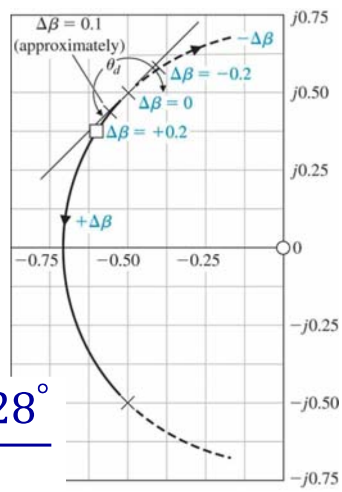
Sensitivity and the Root Locus -4

- Variation of β

$$s^2 + (1+\Delta\beta)s + 0.5 = 0$$

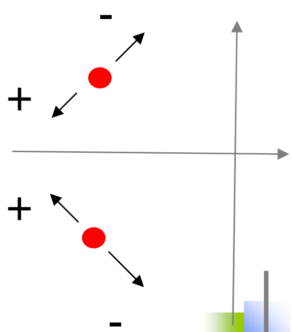
$$1 + \Delta\beta \frac{s}{s^2+s+K} = 0$$

$$20\%\ variation \qquad \Delta\beta = \pm 0.2$$



⇨ $\beta = 1.2 \qquad S_{\beta+}^{s_1} = \frac{\Delta s_1}{\beta/\Delta\beta} = \frac{0.16\angle-128^\circ}{0.2}$

$$= 0.8\angle-128^\circ$$

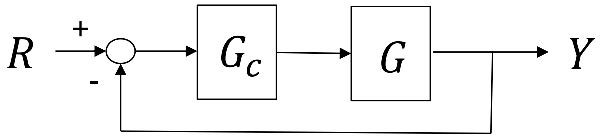


⇨ $\beta = 0.8 \qquad S_{\beta-}^{s_1} = \frac{\Delta s_1}{\beta/\Delta\beta} = \frac{0.125\angle 39^\circ}{0.2}$

$$= 0.625\angle+39^\circ$$

PID Controllers -1

- Form



$$G_c(s) = K_P + \frac{K_I}{s} + K_D s = \frac{K_D s^2 + K_P s + K_I}{s}$$

proportional *integral* *derivative*

Add a pole and two zeros

$$u(t) = K_P e(t) + K_I \int e_I(t) + K_D \frac{de(t)}{dt}$$

Table 7.6 Effect of Increasing the PID Gains K_p, K_D, and K_I on the Step Response

PID Gain	Percent Overshoot	Settling Time	Steady-State Error
Increasing K_P	Increases	Minimal impact	Decreases
Increasing K_I	Increases	Increases	Zero steady-state error
Increasing K_D	Decreases	Decreases	No impact

PID Controllers -2

- PI controller, $K_D = 0$

$$G_c(s) = K_P + \frac{K_I}{s} = \frac{K_P s + K_I}{s}$$

Add a pole and a zero

Low-pass, phase-lag

System type +1, reduce e_{ss}

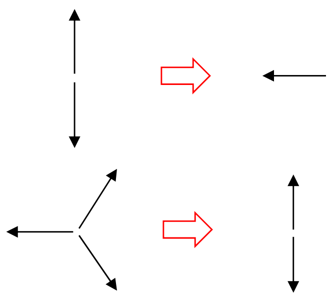
- PD controller, $K_I = 0$

$$G_c(s) = K_P + K_D s$$

Add a zero

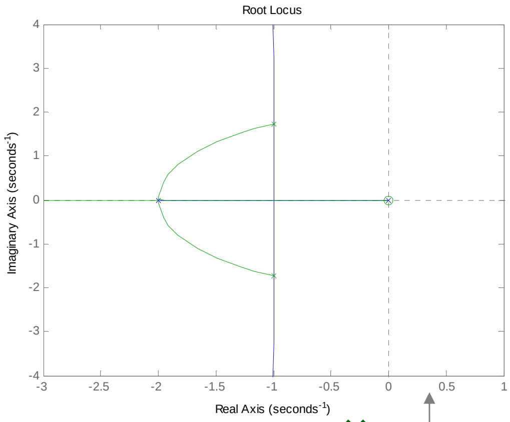
(# of asymptotes -1)

High-pass, phase-lead



Example -1

- Plant $G = \frac{1}{s(s+a)}$



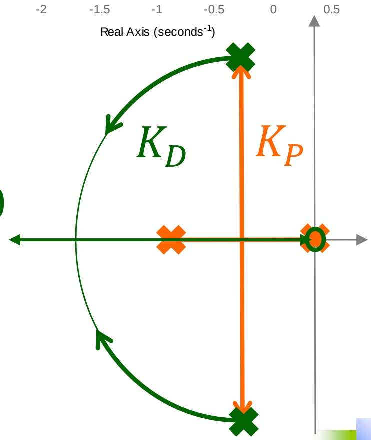
- Using the PD controller

$$G_c = K_P + K_D s$$

$$\Delta(s) = 1 + \frac{K_P + K_D s}{s(s+a)} = 0$$

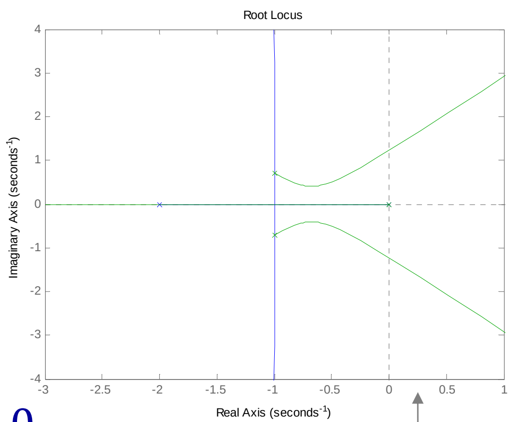
$$set\ K_D = 0 \qquad 1 + K_P \frac{1}{s(s+a)} = 0$$

$$with\ K_D \qquad 1 + K_D \frac{s}{s^2 + as + K_P} = 0$$



Example -2

- Using the PI controller



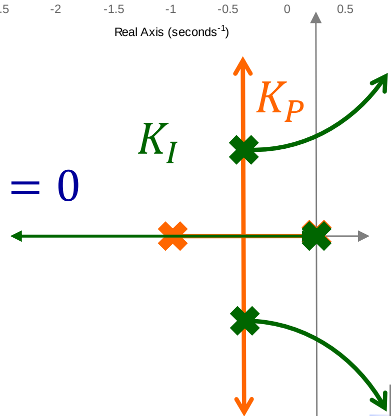
$$G_c = K_P + \frac{K_I}{s}$$

$$\Delta(s) = 1 + \frac{K_P s + K_I}{s} \frac{1}{s(s+a)} = 0$$

$$s^3 + as^2 + K_P s + K_I = 0$$

$$set\ K_I = 0 \qquad 1 + K_P \frac{1}{s(s+a)} = 0$$

$$with\ K_I \qquad 1 + K_I \frac{1}{s(s^2 + as + K_P)} = 0$$



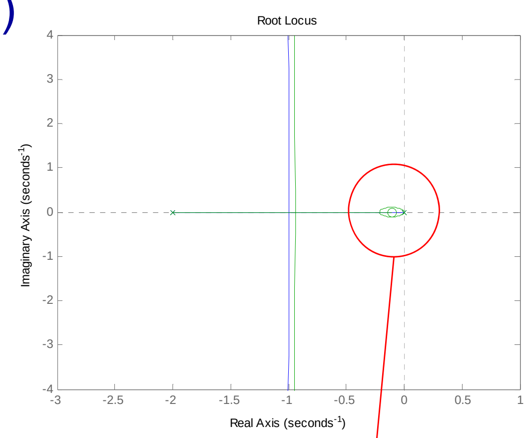
Example -3

- Using the PI controller (2nd method)

$$instead\quad G_c = K_P + \frac{K_I}{s} = \frac{K_P(s + \frac{K_I}{K_P})}{s}$$

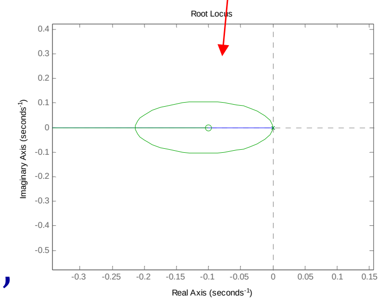
$$choose\ the\ zero, s = -\frac{K_I}{K_p},$$

$$close\ to\ the\ orgin$$



$$then\ varying\ K_P$$

$$1 + K_P\frac{(s + \frac{K_I}{K_P})}{s^2(s+a)} = 0$$



$$Shape\ of\ the\ R.L.\ only\ slightly\ changes,$$

$$but\ the\ system\ type + 1$$

PID Tuning

- Closed-loop Ziegler-Nichols tuning method

Table 7.7 Ziegler-Nichols PID Tuning Using Ultimate Gain, K_U, and Oscillation Period, P_U

Ziegler-Nichols PID Controller Gain Tuning Using Closed-loop Concepts

Controller Type	K_P	K_I	K_D
Proportional (P) $G_c(s) = K_P$	$0.5K_U$	–	–
Proportional-plus-integral (PI) $G_c(s) = K_P + \frac{K_I}{s}$	$0.45K_U$	$\frac{0.54K_U}{T_U}$	–
Proportional-plus-integral-plus-derivative (PID) $G_c(s) = K_P + \frac{K_I}{s} + K_D s$	$0.6K_U$	$\frac{1.2K_U}{T_U}$	$\frac{0.6K_U T_U}{8}$

Negative Gain Root Locus -1

Table 7.9 Seven Steps for Sketching a Negative Gain Root Locus (color text denotes changes from root locus steps in Table 7.2)

Step	Related Equation or Rule
1. Prepare the root locus sketch.	
(a) Write the characteristic equation so that the parameter of interest, K, appears as a multiplier.	(a) $1 + KP(s) = 0$
(b) Factor $P(s)$ in terms of n poles and M zeros	(b) $1 + K\dfrac{\prod_{i=1}^{M}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0$
(c) Locate the open-loop poles and zeros of $P(s)$ in the s-plane with selected symbols.	(c) $\times$ = poles, ○ = zeros
(d) Determine the number of separate loci, SL.	(d) Locus begins at a pole and ends at a zero. $SL = n$ when $n \geq M$; n = number of finite poles, M = number of finite zeros.
(e) The root loci are symmetrical with respect to the horizontal real axis.	

Negative Gain Root Locus -2

Table 7.9 (continued) Seven Steps for Sketching a Negative Gain Root Locus (color text denotes changes from root locus steps in Table 7.2)

Step	Related Equation or Rule
2. Locate the segments of the real axis that are root loci.	Locus lies to the left of an even number of poles and zeros.
3. The loci proceed to the zeros at infinity along asymptotes centered at σ_A and with angles ϕ_A.	$\sigma_A = \dfrac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{M}(-z_i)}{n - M}$. $\varphi_A = \dfrac{2k}{n-M}180°, k = 0, 1, 2, \ldots (n - M - 1)$
4. Determine the points at which the locus crosses the imaginary axis (if it does so).	Use Routh-Hurwitz criterion (see Section 6.2).
5. Determine the breakaway point on the real axis (if any).	a) Set $K = p(s)$ b) Determine roots of $dp(s)/ds = 0$ or use graphical method to find maximum of $p(s)$.
6. Determine the angle of locus departure from complex at or poles and the angle of locus arrival at complex zeros using the phase criterion.	$\angle P(s) = \pm k360°$ at $s = -p_j$ or $-z_i$
7. Complete the negative gain root locus sketch.	

Negative Gain Root Locus -3

- Example $1 + K\frac{s-20}{s^2+5s-50} = 0$

$$1 + K\frac{s-20}{s^2+5s-50} = 0$$

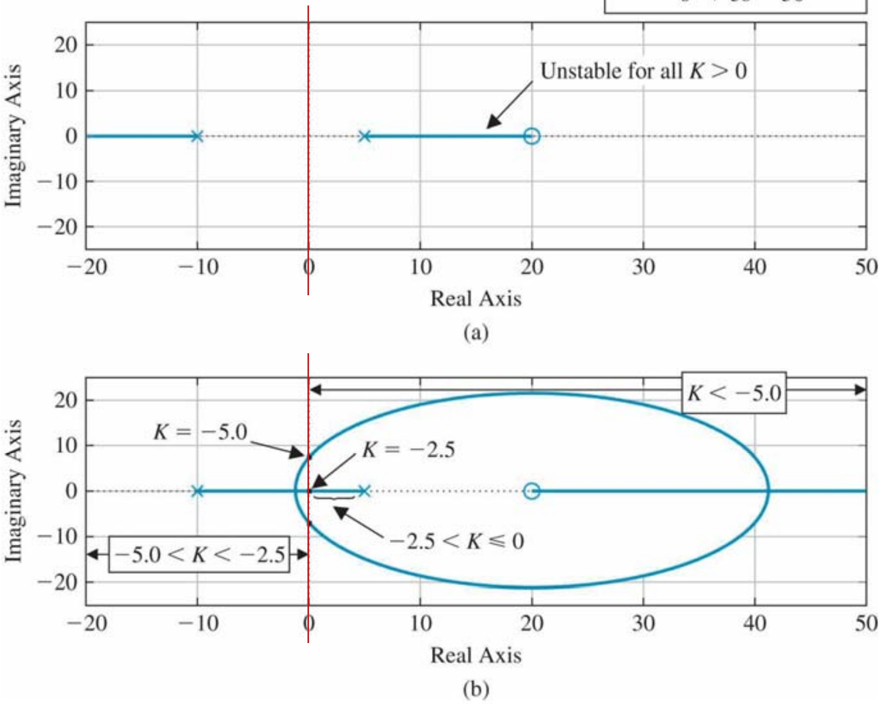


Figure 7.44 (a) Root locus for $0 \le K < \infty$. (b) Negative gain root locus for $-\infty < K \le 0$.

The End

- Questions?