Chap 6 The Stability of Linear Feedback Systems

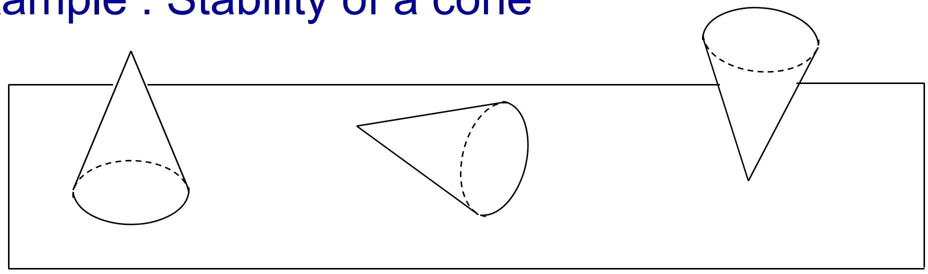
林沛群
國立台灣大學
機械工程學系

The Concept of Stability -1

- Definitions
  - A stable system is a dynamic system with a bounded response to a bounded input (BIBO stability)
  - A linear system is stable if and only if the absolute value of its impulse response, g(t), integrate over an infinite range is finite

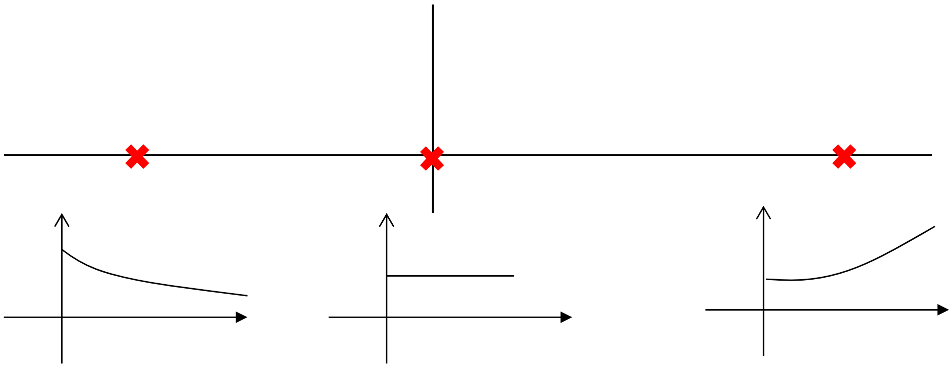
The Concept of Stability -2

- Example : Stability of a cone



stable　　neutral　　unstable

- Example: Stability of dynamic systems



The Concept of Stability -3

- Stability (transfer function)

$$T(s) = \frac{p(s)}{q(s)} = \frac{k\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+\sigma_k)\prod_{m=1}^{R}(s^2+2\alpha_m s+(\alpha_m^2+\omega_m^2))}$$

$$q(s) = \Delta(s) = 0 \quad roots\ \ are\ "poles"$$

$$p = -\sigma_k, -\alpha_m \pm \omega_m$$

$$The\ impulse\ response\ of\ T(s), (N=0)$$

$$y(t) = \sum_{k=1}^{Q} A_k e_k^{-\sigma_k t} + \sum_{m=1}^{R} B_m\left(\frac{1}{\omega_m}\right)e^{-\sigma_m t}\sin(\omega_m t+\theta_m)$$

$$A_k = constant = f(z_i, \sigma_k, \alpha_m, \omega_m, k)$$

$$B_m = constant = g(z_i, \sigma_k, \alpha_m, \omega_m, k)$$

The Concept of Stability -4

- For a system to be stable (necessary and sufficient)

  ⇨ All poles of the transfer function have negative real parts (in LHP)

"distinct" roots

- If the system has simple roots on the imaginary axis

  ⇨ Marginally stable: only certain bounded inputs will cause the output to become unbounded

The Routh-Horwitz Stability Criterion -1

- Motivation
  - Can we know stability of the linear system without finding the locations of all poles?

$$\Delta(s) = q(s) = a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

$$(a_n > 0)$$

  - The answer is YES!

The Routh-Horwitz Stability Criterion -2

- Necessary Condition

$$q(s) = a_n(s - r_1)(s - r_2) \dots (s - r_n) = 0$$

$$= \boxed{\begin{aligned} & a_n s^n - (r_1 + r_2 + \cdots + r_n)s^{n-1} \\ & +(r_1 r_2 + r_1 r_3 + \cdots + r_{n-1} r_n)s^{n-2} \\ & -(r_1 r_2 r_3 + r_1 r_2 r_4 + \cdots + r_{n-2} r_{n-1} r_n)s^{n-3} \\ & \vdots \\ & +(-1)^n r_1 r_2 \dots r_n \end{aligned}} = 0$$

- If all the roots are in the LHP
  - All the coefficients of the polynomial MUST have the same sign
  - None of the coefficients vanishes

⇨ This condition is NOT sufficient

The Routh-Horwitz Stability Criterion -3

- Routh's Tabulation

$$a_6 s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

as the example

first column

s^6	a_6	a_4	a_2	a_0
s^5	a_5	a_3	a_1	0
s^4	$\frac{a_4 a_5 - a_3 a_6}{a_5} = A$	$\frac{a_2 a_5 - a_1 a_6}{a_5} = B$	$\frac{a_0 a_5 - a_6 \cdot 0}{a_5} = a_0$	0
s^3	$\frac{a_3 A - a_5 B}{A} = C$	$\frac{a_1 A - a_0 a_5}{A} = D$	$\frac{A \cdot 0 - a_5 \cdot 0}{A} = 0$	0
s^2	$\frac{BC - AD}{C} = E$	$\frac{a_0 C - A \cdot 0}{C} = a_0$	$\frac{C \cdot 0 - A \cdot 0}{C} = 0$	0
s^1	$\frac{DE - a_0 C}{E} = F$	0	0	0
s^0	$\frac{a_0 F - E \cdot 0}{F} = a_0$	0	0	0

The Routh-Horwitz Stability Criterion -4

- Sufficient condition: The Routh-Horwitz Criterion
  - The number of roots of $q(s)$ with positive real parts

    is equal to

    the number of changes in sign of the first column

    of the Routh Tabulation

⇨ For a stable system,
NO CHANGE IN SIGN in the first column of Routh's Tabulation

Routh's Tabulation -1

- Case 1: No element in the first column is zero
- Example 1: 2nd–order system

$$q(s) = a_2 s^2 + a_1 s + a_0 \qquad a_2 > 0$$

$$\begin{array}{l|c|c} s^2 & a_2 & a_0 \\ s^1 & a_1 & 0 \\ s^0 & \dfrac{a_0 a_1 - a_2 \cdot 0}{a_1} = a_0 & 0 \end{array}$$

⇨ Stable : a_0, a_1, a_2 are all positive

Necessary condition = sufficient condition

Routh's Tabulation -2

- Example 2: 3rd–order system

$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0 \qquad a_3 > 0$$

s^3	a_3	a_1
s^2	a_2	a_0
s^1	$\frac{a_1 a_2 - a_0 a_3}{a_2} = A$	0
s^0	$\frac{A a_0 - a_2 \cdot 0}{A} = a_0$	0

⇒ Stable : Necessary: a_0, a_1, a_2, a_3 positive

Sufficient: $A > 0 \rightarrow a_1 a_2 > a_0 a_3$

so $a_1 > \frac{a_0 a_3}{a_2}$

條件強於 $a_1 > 0$

Routh's Tabulation -3

- Case 2 : A zero in the first column, and other elements of the same row are nonzero.

⇒ Replace 0 with a small positive ϵ and complete the tabulation

- Example 3 : $q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10$

s^5	1	2	11
s^4	2	4	10
s^3	~~0~~ → ϵ	6	0
s^2	$\frac{4\epsilon - 12}{\epsilon} \sim -\frac{12}{\epsilon} = A < 0$	10	0
s^1	$\frac{6A - 10\epsilon}{A} \sim 6$	0	0
s^0	10	0	0

Roots:
0.8950 + 1.4561i
0.8950 - 1.4561i
-1.2407 + 1.0375i
-1.2407 - 1.0375i
-1.3087

⇒ Two sign changes, 2 roots in RHP, unstable

Routh's Tabulation -4

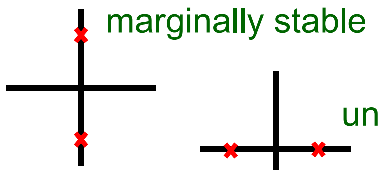
- Example 4 : $q(s) = s^4 + s^3 + s^2 + s + K \quad (K \neq 0)$

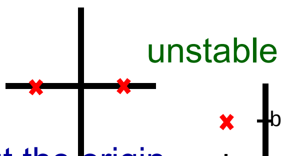
s^4	1	1	K
s^3	1	1	0
s^2	$0 \to \epsilon$	K	0
s^1	$\frac{\epsilon - K}{\epsilon} \sim -\frac{K}{\epsilon}$	0	0
s^0	K	0	0

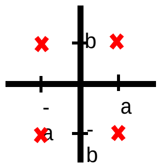
⇨ Unstable for all values of K

Routh's Tabulation -5

- Case 3: A row of zeros
  - When does this happen?
    - $(s + j\omega)(s - j\omega) \geq 1\ pair$
    - $(s + \sigma)(s - \sigma) \geq 1\ pair$
    - Complex-conjugate roots forming symmetry about the origin







  - Solving steps
    - Form the auxiliary equation U(s)=0 by using the coefficients from the row just preceding the row of zeros
    - $\frac{dU}{ds} = 0$
    - Replace the zeros with coefficients of $\frac{dU}{ds} = 0$
    - Continue with Routh's tabulation

Routh's Tabulation -6

- Example 5: $q(s) = s^3 + 2s^2 + 4s + k$

s^3	1	4
s^2	2	k → 8
s^1	$\frac{8-k}{2}$ → 0 → 4	0
s^0	k → 8	0

The 3[rd] root is in LHP

$when\ k = 8$

$U(s) = 2s^2 + 8s^0 = 2s^2 + 8 = 2(s^2+4)$
$= 2(s+j2)(s-j2)$

$\frac{dU(s)}{ds} = 4s$

$a\ factor\ of\ \Delta(s) = 0$

$roots \pm j2$

$if\ 0 < k < 8 \rightarrow stable$

$if\ k = 8 \rightarrow marginally\ stable$

Note: $q(s) = (s+2)(s+j2)(s-j2)$

Routh's Tabulation -7

- Example 6: $\Delta(s) = s^5 + 2s^4 + 5s^3 + 10s^2 + 4s + 8$

s^5	1	5	4
s^4	2	10	8
s^3	0 → 8	0 → 20	0
s^2	5	8	0
s^1	7.2	0	
s^0	8	0	

The 5[th] root is in LHP

$U = 2s^4 + 10s^2 + 8 = 2(s^4 + 5s^2 + 4)$
$= 2(s^2+1)(s^2+4)$

$\frac{dU}{ds} = 8s^3 + 20s$

$\rightarrow marginally\ stable$

Note: $q(s) = (s+2)(s+j)(s-j)(s+2j)(s-2j)$

Routh's Tabulation -8

- Example 7: $q(s) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1$

s^5	1	2	1
s^4	1	2	1
s^3	~~0~~ 4	~~0~~ 4	0
s^2	1	1	0
s^1	~~0~~ 2	0	
s^0	1	0	

$U = s^4 + 2s^2 + 1 = (s^2+1)^2$ *repeated roots on* $j\omega - axis$ $\rightarrow$ *unstable*

$\frac{dU}{ds} = 4s^3 + 4s + 0$

$U = s^2 + s^0 = s^2 + 1$

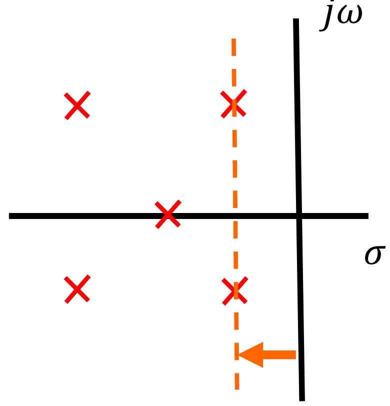
$\frac{dU}{ds} = 2s$

The 5th root is in LHP

Note: $q(s) = (s+1)(s+j)(s-j)(s+j)(s-j)$

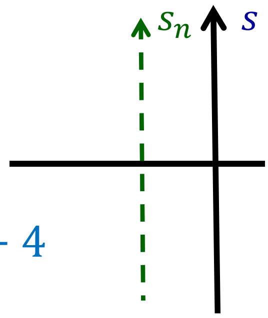
The Relative Stability -1

- Motivation
  - If the system is absolute stable, then we can think about "how stable it is" → Relative stability
  - By axis-shifting, we can know how far the original system to the unstable margin



The Relative Stability -2

- Example



$q(s) = s^3 + 4s^2 + 6s + 4$

$4 \times 6 - 1 \times 4 = 20 > 0$ stable

$try\ s_n = s + 2$

$q(s_n) = (s_n - 2)^3 + 4(s_n - 2)^2 + 6(s_n - 2) + 4$

$= s_n^3 - 2s_n^2 + 2s_n + 0$ unstable

$try\ s_n = s + 1$

$q(s_n) = (s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4$

$= s_n^3 + s_n^2 + s_n + 1$ $1 \times 1 - 1 \times 1 = 0$

s^3	1	1
s^2	1	1
s^1	2	0

$U(s) = s_n^2 + 1 = (s_n + j)(s_n - j)$

$\frac{dU(s)}{ds_n} = 2s_n$ $\rightarrow marginally\ stable$

Note: $q(s_n) = ({s_n}^2 + 1)(s_n + 1)$

The stability of State Variable Systems

- How to utilize Routh-Hurwitz Criterion
  - In transfer function $\rightarrow$ $q(s) = \Delta(s) = 0$
  - In signal-flow graph $\rightarrow$ $Mason's\ formula\ to\ get\ \Delta$
  - In block diagram $\rightarrow$ $block\ diagram\ reduction$
  - In state space $\rightarrow$ $det(sI - A)^{-1}$

The End

- Questions?